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November 25, 2002

Algebraic Topology, No. 4
WS 02/03

Sketches of solutions to selected problems

1. Let $\mathcal{K}$ be a simplicial complex with given orientation.

   Then $c\mathcal{K}$ consists of the simplices in $\mathcal{K}$ with the given orientation and the oriented simplices $a\sigma^q = [a, a_0, \ldots, a_q]$, for all positively oriented simplices $\sigma^q = [a_0, \ldots, a_q]$ from $\mathcal{K}$.

   Now we have (letting $\sigma^{q-1} = [a_0, \ldots, a_{q-1}] \in \mathcal{K}$)

$$\begin{aligned}
\partial a\sigma^{q-1} &= \partial[a, a_0, \ldots, a_{q-1}] = [a_0, \ldots, a_{q-1}] + \sum_{i=0}^{q-1}(-1)^{i+1}[a, a_0, \ldots, \hat{a}_i, \ldots, a_{q-1}] \\
&= \sigma^{q-1} - \sum_{i=0}^{q-1}(-1)^{i}[a, a_0, \ldots, \hat{a}_i, \ldots, a_{q-1}] \\
&= \sigma^{q-1} - a\sum_{i=0}^{q-1}(-1)^{i}[a_0, \ldots, \hat{a}_i, \ldots, a_{q-1}] \\
&= \sigma^{q-1} - a\partial\sigma^{q-1}
\end{aligned}$$

2. Clearly, both sides of the equation in (1) extend linearly from simplices to chains, so we obtain (2).

3. Every simplex in $c\mathcal{K}$ has a unique representation as $[a_0, \ldots, a_q]$ with $a_0 < \cdots < a_q$. If $a_0 = 0 = a$, we have $[a_1, \ldots, a_q] \in \mathcal{K}$ (why?). This yields the decomposition given in (3). We obtain uniqueness because for every simplex $\sigma \in c\mathcal{K}$ containing a, there is exactly one simplex $\sigma' \in \mathcal{K}$ such that $\sigma = a\sigma'$.

4. We have

$$0 = \partial c^q = \partial a d^{q-1} + \partial e^q \underset{(2)}{=} d^{q-1} - a\partial d^{q-1} + \partial e^q$$

   By (3), we get $d^{q-1} + \partial e^q = 0$, or equivalently $d^{q-1} = -\partial e^q$ [†]. Substituting this in the formula for c^q, we obtain

$$c^q = a(-\partial e^q) + e^q = e^q - a\partial e^q \underset{(2)}{=} \partial a e^q$$

[†] We also obtain $\partial d^{q-1} = 0$, but this follows from the other equation as well.

5. Now $H_q(c\mathcal{K}) = 0$ follows ($q \geq 1$), because we have just shown that every q-cycle is a q-boundary.

6. A q-simplex $[a_0, \ldots, a_q]$ can be viewed as the cone over the $q-1$-dimensional complex consisting of all faces of a $q-1$-dimensional simplex (compare the sketches on the exercise sheets!). Hence, (5) applies and the claim follows.

7. For $q \geq 2$, we have $q-1 \geq 1$, so the corresponding part of the long exact homology sequence of the pair $(c\mathcal{K}, \mathcal{K})$ is

$$\cdots \longleftarrow \underbrace{H_{q-1}(c\mathcal{K})}_{=0} \longleftarrow H_{q-1}(\mathcal{K}) \xleftarrow{\partial^*} H_q(c\mathcal{K}, \mathcal{K}) \longleftarrow \underbrace{H_q(c\mathcal{K})}_{=0} \longleftarrow \cdots$$

   This implies that ∂^* is an isomorphism (it is injective by the 0 on the right and surjective by the 0 on the left).

(Additional:) Try to find all the places where we used $q \geq 1$, and make sure you understand why our calculations do not generalize to $H_q(c\mathcal{K}) = \{0\}$ even for $q = 0$.