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Sketches of solutions to selected problems

1. The proof works precisely as the proof of theorem 5.3.3.

2. b) This is immediate from (a), 7.4.6 and 7.4.4.

3. a) An orientable closed surface of genus g (i.e. $\underbrace{T\#\cdots\#T}_{g-\text{times}}$) can be described by the word $\prod_{j=1}^{g} a_j b_j a_j^{-1} b_j^{-1}$. Hence, we have one 2-cell and $2g$ 1-cells. Drawing a sketch, we see that all vertices of the polygon with edges marked by the word $\prod_{j=1}^{g} a_j b_j a_j^{-1} b_j^{-1}$ are identified. Hence, there is only one 0-cell.

   b) The non-orientable closed surfaces are of the form $\underbrace{P\#\cdots\#P}_{g-\text{times}}$, and they can be represented by the word $\prod_{j=1}^{g} a_j a_j$. Hence, we have one 2-cell and g 1-cells. As above, we obtain one 0-cell only.

   c) Real projective n-space P^n is the set of lines through the origin in $\mathbb{R}^{n+1}$. Topologically, we imagine it as $S^n/\pm 1$, where the relation is $x \sim -x$.[†] To obtain P^n as a cell complex, we view it as the upper hemisphere of S^n (denoted by S^n_+ before). Now the interior of S^n_+ is an n-cell (and no points of the interior are equivalent to any other point of S^n_+). The boundary of S^n_+ is S^{n-1}, and the relation is given by $x \sim -x$.

   So the cell decomposition of P^n is one n-cell, attached to P^{n-1}. Inductively, this shows that P^n consists of one cell in each dimension up to n.

   d) A simplicial complex can clearly be viewed as a cell complex (check the details!).

4. To compute the homology groups of a cell complex X, we only need to understand how the "gluing map" attaches the boundary of an n-cell to the $n-1$ skeleton X^{n-1} of the complex.

   For a closed orientable surface X_g of genus g, we obtain $H_0(X_g) = \mathbb{Z} = H_2(X_g)$ (since X_g is connected, and the boundary of the 2-cell is 0). Further, we have $H_1(X_g) = \mathbb{Z}^{2g}$, since the one-skeleton is the one-point union of $2g$ circles, and there are no relations from the 2-cell to be considered.

[†]Since multiplication by -1 is an isometry of $\mathbb{R}^{n+1}$ which leaves the submanifold S^n invariant, P^n can even be viewed as the quotient of the manifold S^n by a group of isometries.

For the non-orientable surface U_g, similar considerations show $H_0(U_g) = \mathbb{Z}$ and $H_2(U_g) = 0$, and finally $H_1(U_g) = \mathbb{Z}^{g-1} \times \mathbb{Z}_2$.

For real projective n-space P^n (where $n \geq 2$), we have $H_0(P^n) = \mathbb{Z}$ and $H_1(P^n) = \mathbb{Z}_2$ (by 7.4.8).

To compute the higher dimensional homology groups, we use Theorem 7.4.6: By 7.4.4, we have $H_q((P^n)^q, (P^n)^{q-1}) = \mathbb{Z}$ for $1 \leq q \leq n$. For the q-cell γ^q (for $1 \leq q \leq n$), we have $\partial\gamma^q = 2\gamma^{q-1}$ if q is even and $\partial\gamma^q = 0$ if q is odd; this follows from our discussion of the structure of P^n as a cell complex, and the fact that the map $x \mapsto -x$ of $\mathbb{R}^q$ is orientation reversing if and only if q is odd. Thus we obtain:

$$H_q(P^n) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}_2 & 1 \leq q < n \text{ and } q \text{ odd} \\ \mathbb{Z} & q = n \quad \text{and } q \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$