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Sketches of solutions to selected problems

1. a) We want to show that $-f$ is homotopic to g. We try the standard procedure: connecting $g(x)$ to $-f(x)$ with a line segment, and projecting the segment to S^n via normalization.[†]

   This is well-defined if the segment joining $g(x)$ and $-f(x)$ does not contain the origin. This is true, since a segment joining points on S^n through the origin must connect antipodal points; this cannot happen since $f(x) \neq g(x)$.

   Setting $l_t(x) := tg(x) - (1-t)f(x)$, the formula for the homotopy is $H_t(x) = l_t(x)/\left\|l_t(x)\right\|$.

   b) If $-f \sim g$, then $\deg(f) = (-1)^{n+1}\deg(g)$ (by 7.3. and 7.4). Hence, if $|\deg(f)| \neq |\deg(g)|$, the assumption of (a) has to be wrong and we conclude that there exists a point x such that $f(x) = g(x)$.

   c) By (a), f is homotopic to g if $f(x) \neq -g(x)$ holds for all points in S^n.

2. We consider the following setting, which is slightly more general than problem 2 on the sheet, but less general than exercise 3:

   Let B be a compact convex subset of a hyperplane $H \subset \mathbb{R}^n$ with $0 \notin H$. Assume that B has nonempty interior (as a subset of H).

   Let φ be a linear map $\mathbb{R}^n \to \mathbb{R}^n$ which satisfies $\varphi(B) \subset \mathbb{R}_{\geq 0} \cdot B =: K$.

   We show that φ has an eigenvector with non-negative eigenvalue in B. If $0 \in \varphi(B)$, we are done. Otherwise, we can project $\varphi(x)$ to B along the ray $\mathbb{R}_{\geq 0} \cdot \varphi(x)$ and obtain a continuous map $\vartheta : B \to B$ in this way. (Observe that ϑ is indeed uniquely defined and continuous.)

   Since B is homeomorphic to $\bar{B}^{n-1}$ by exercise 1.3, we know that ϑ has a fixed point by the Brouwer fixed point theorem. This fixed point is an eigenvector of φ with non-negative eigenvector.

3. First, let us generalize the previous exercise by dropping the assumption that B has nonempty interior (as a subset of H):

   Let m be the maximal number such that there are m linearly independent vectors in B, and let $v_1, \ldots, v_m$ be m linearly independent vectors in B.

   Then B is contained in the m-plane H' in $\mathbb{R}^n$ spanned by the v_i, and B has nonempty interior as a subset of H' (because the convex hull of $v_1, \ldots, v_m$ is contained in B), showing that B is homeomorphic to $\bar{B}^m$. Now the arguments work as before.

[†] Observe that this means moving $g(x)$ along the great circle through $g(x)$ and $-f(x)$; this great circle is uniquely determined if $g(x)$ and $-f(x)$ are not antipodal points.

Now we want to reduce problem 3 to the previous one.

To this end, let $p \in B$ be a point such that $d(0, B) = d(0, p) > 0$ (this is possible since B is compact).

We want to show $B \subset \{x \in \mathbb{R}^n \mid \langle x, p\rangle \geq \langle p, p\rangle\}$.

Let $H := \{x \in \mathbb{R}^n \mid \langle x, p\rangle = \langle p, p\rangle\}$ be the hyperplane orthogonal to p containing p.

In words, we claim "B lies in the halfspace determined by H not containing 0."

On the contrary assume that there is a point $x \in B$ in the same halfspace as 0. Then the segment $[p, x]$ is contained in B, which contradicts the choice of p (because points on $[p, x]$ close to p have shorter distance from the origin).

Thus, we can project B to $B' \subset H$ along rays through the origin. Note that B' is compact and convex. Hence, the previous exercise shows existence of an eigenvector v in B', and every point in $B \cap (\mathbb{R}_{\geq 0} \cdot v)$ is an eigenvector of φ with nonnegative eigenvalue.

4. We can view S^3 as the unit sphere in the space of quaternions. The tangent space at 1 corresponds exactly to the purely imaginary quaternions. Thus, we can view i, j, k as a basis for the tangent space at 1. For an arbitrary point $p \in S^3$, define $v_1 = pi, v_2 = pj, v_3 = pk$. Then v_1, v_2, v_3 are linearly independent continuous vector fields on S^3 (details are left to the reader).

5. If $n = 1$, the claim can easily be verified. We assume $n \geq 2$.

   First, we define $g := f / \|f\| : B^n \to S^{n-1}$. We want to show that the restriction of g to S^{n-1} has degree 0.

   Now $g_* : 0 \cong H_{n-1}(B^n) \to H_{n-1}(S^{n-1}) \cong \mathbb{Z}$ is trivial and hence, so is $(g|_{S^{n-1}})_*$ (because $g|_{S^{n-1}} = g \circ i$ where i is the embedding).

   By 7.5, the claim follows since $0 = \deg(g|_{S^{n-1}}) \neq \pm 1$.

6. We show the claim by induction:

   We know that the result holds for $n = 1$.

   So let $f_k^{n-1} : S^{n-1} \to S^{n-1}$ be a map of degree k. We want to define a map $f_k^n : S^n \to S^n$ of degree k.

   We use $S^n \cong (S^{n-1} \times [-\pi, \pi])/\{(S^{n-1}, -\pi), (S^{n-1}, \pi)\}$,
   and define f_k^n by $f_k^n(p, t) = (f_k^{n-1}(p), t)$. Observe that f_k^n is well-defined and continuous (check that; in particular at the poles).

   Observe that f_k^n restricted to the equator is f_k^{n-1}.

   With arguments exactly as in the proof of 7.4.i (on the extra sheet of the lecture notes), we see that $(f_k^n)_*$ acts on $H_n(S^n)$ in the same way as $(f_k^{n-1})_*$ acts on $H_{n-1}(S^{n-1})$. (In particular, we consider the same diagram!)