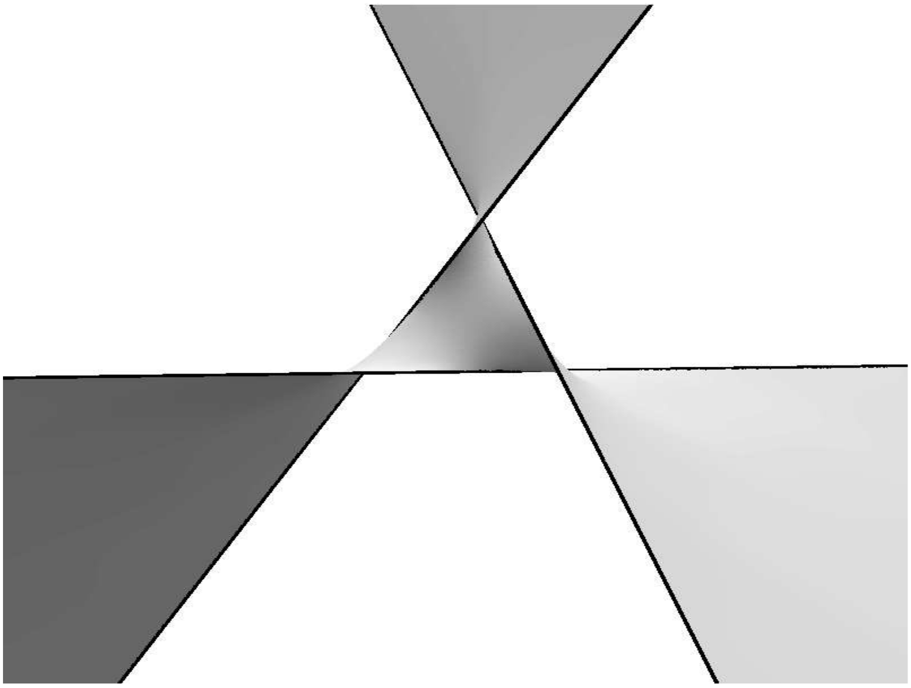
On Trinoids and Minimal Disks bounded by Lines



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Chapter 0

Introduction

In this thesis, we examine geometric properties of certain surfaces of constant mean curvature (CMC). Readers not familiar with surfaces of this kind may view Theorems I.1.1 and I.5.3 as definitions (of the types of surfaces relevant here).

Some of the surfaces we consider will have hyperbolic 3-space $\mathbb{H}^3$ as ambient space; a (very) brief introduction to hyperbolic space is given in section I.4.

The focus of this text is on *trinoids*, which are surfaces of constant mean curvature 1 in $\mathbb{H}^3$, with three ends and genus zero.[1]

On the last pages, the reader can find an index which lists where certain expressions are introduced.

In the first chapter, we present the background material necessary to follow the arguments used in the other chapters. Most Theorems are stated without proof, but with precise reference.

In section I.2, we introduce the associate construction, relating minimal surfaces in $\mathbb{R}^3$ to one another.

Section I.5 discusses the Bryant cousin relation, which relates CMC-1 surfaces in $\mathbb{H}^3$ to minimal surfaces in Euclidean 3-space $\mathbb{R}^3$; this is a special case of Lawson's correspondence, described in section I.6.

In section I.7, we examine how symmetries of CMC-1 surfaces in $\mathbb{H}^3$ and symmetries of the corresponding minimal surfaces in $\mathbb{R}^3$ are related.[2]

In the second chapter, we examine minimal surfaces bounded by lines or rays. In particular, we obtain information on the asymptotic behavior of

[1] The precise definition of the class of trinoids we will use is given as definition III.1.2.

[2] We discuss a Theorem from an article with a small mistake, show the correct version, and generalize slightly.

minimal ends bounded by rays that are contained in a convex wedge of a slab. The proof relies on a particular way of applying the maximum principle to make statements about minimal surfaces contained in a wedge of a half-space.

Furthermore, we show that if a “standard sequence” of solutions to the Plateau problem (in a convex wedge of a slab) consists of embedded minimal disks, then the sequence satisfies uniform local area bounds, which implies that the sequence of surfaces converges to a limit surface with asymptotic behavior of the kind discussed before.

In the third chapter, we turn to the study of trinoids.

After cutting a trinoid in “two halves”, we use the Bryant cousin relation, composed with the associate construction, to obtain a minimal disk bounded by three lines.

The asymptotics of an end of a trinoid can be described by a real parameter (in $(0,\infty)\setminus\{\pi\}$). Since a trinoid has three ends, we obtain a triple of parameters for every trinoid. Examining the constellation of boundary lines of the conjugate minimal disk bounded by three lines, we obtain a necessary condition on the triple. Further, using the results from chapter 2, we show that trinoids for certain parameter-triples exist.

A personal note:
I would like to cordially thank my advisor Prof. Karsten Große-Brauckmann for the vast amount of time he spent discussing the (future) contents of this thesis with me.

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Chapter I

Preliminaries

We start with some definitions:

Definition I.0.1. We define a *CMC-H surface* to be a surface of constant mean curvature H. In this thesis, the important cases are $H = 0$ and $H = 1$.

Definition I.0.2. A *minimal surface* is a CMC-0 surface (usually, the minimal surfaces considered in this thesis are in $\mathbb{R}^3$).

Definition I.0.3. A minimal surface M (in $\mathbb{R}^3$) is of *finite topology* if it can be represented as a map $\Sigma \backslash \{p_j \,|\, 1 \leq j \leq k\} \to \mathbb{R}^3$, where Σ is a compact Riemann surface (and the points p_j are distinct).

Then M has k ends, and an end is the restriction of M to a closed simply connected neighborhood U of some p_{j_0}, such that U does not contain p_j for $j \neq j_0$.

If M is *properly immersed*, every sequence in Σ converging to some p_{j_0} is mapped to a sequence converging to $\infty \in \hat{\mathbb{R}}^3$ (in the topology of the one-point compactification).

Observe that a properly immersed minimal surface M of finite topology in $\mathbb{R}^3$ is complete (with respect to the induced metric on $\Sigma \backslash \{p_j\}$).

Definition I.0.4. In this thesis, when we speak of an *immersed disk*, we mean an immersion of $\bar{D} \backslash \{p_j \,|\, 1 \leq j \leq k\}$ into some space, where D is the open unit disk and the p_j are distinct points in ∂D.

I.1 Weierstrass Representation for Minimal Surfaces in $\mathbb{R}^3$

Minimal surfaces in $\mathbb{R}^3$ with finite topology and finite total curvature can be represented in the following "standard way":

Theorem I.1.1 (Weierstrass Representation). *Let Σ be a compact Riemann surface, and $\{p_j \,|\, 1 \le j \le k\} \subset \Sigma$ be a finite number of points (representing the ends of the minimal surface defined in this Theorem). Fix a point $z_0 \in \Sigma \backslash \{p_j\}$. Let g be a meromorphic function on Σ, and f a holomorphic function on $\Sigma \backslash \{p_j\}$. Assume that, for any point in $\Sigma \backslash \{p_j\}$, f has a zero of order $2k$ if and only if g has a pole of order k, and that f has no other zeroes on $\Sigma \backslash \{p_j\}$. Then, in terms of local holomorphic coordinates z on $\Sigma \backslash \{p_j\}$,*

$$\Phi_W(z) := Re \int_{z_0}^{z} \begin{pmatrix} (1-g^2) f d\xi \\ i(1+g^2) f d\xi \\ 2 g f d\xi \end{pmatrix} \tag{I.1.1}$$

is a conformal minimal immersion of the universal cover $\widetilde{\Sigma \backslash \{p_j\}}$ of $\Sigma \backslash \{p_j\}$ into $\mathbb{R}^3$. If Φ_W is well-defined on $\Sigma \backslash \{p_j\}$, then it has finite total curvature. Furthermore, any complete minimal surface with finite total curvature in $\mathbb{R}^3$ can be represented this way.

Proof. This is the version of the Weierstrass Theorem as stated in [Ross01, Thm. 2]. Note that some authors use an additional factor of 1/2 (their surface is $\Phi_W/2$). In example I.1.6, we will see why. However, our version works more nicely with the Bryant representation explained in section I.5.

For a proof, we refer the reader to [Oss86, §8] □

Remark I.1.2. We deduce from the Theorem above that finite total curvature implies finite topology. The function g is the stereographic projection of the Gauss map of Φ_W. In particular, we observe that for a surface of finite total curvature, the Gauss map extends meromorphically to the ends (i.e. for every end, there is a well-defined limit normal). ♢

Definition I.1.3. The pair $(g, f dz)$ is called the *Weierstrass data* of the minimal surface Φ_W.

The Weierstrass data is usually denoted by (g, ω).

Remark I.1.4. The presentation of minimal surfaces by Weierstrass data as in formula I.1.1 works for all simply connected minimal surfaces (cf. [Oss86, Lemmas 8.1,8.2]. However, we do not always obtain a compact Riemann surface Σ (with or without boundary) with the properties from the Theorem if the minimal surface is not complete, does not have finite total curvature, or has boundary. $\diamondsuit$

Lemma I.1.5. *The Riemannian metric induced on $\widetilde{\Sigma\backslash\{p_j\}}$ by Φ_W is $ds^2 = (1+g\bar{g})^2 f\bar{f}dz^2$*

Proof. This is a well-known fact. We shall give a proof by direct calculation. Since $\Phi := \Phi_W$ is conformal, it suffices to compute:

$$\left\langle \frac{\partial\Phi}{\partial u}, \frac{\partial\Phi}{\partial u} \right\rangle = (1+|g|^2)^2|f|^2$$

Of course, one could check conformality by a similar direct calculation. So let $g = u_1 + iv_1$, $f = u_2 + iv_2$. We compute

$$\begin{aligned}
(1-g^2)f &= u_2 - u_1^2u_2 + v_1^2u_2 + 2u_1v_1v_2 + i(v_2 - u_1^2v_2 + v_1^2v_2 - 2u_1u_2v_1)\\
(1+g^2)f &= u_2 + u_1^2u_2 - v_1^2u_2 - 2u_1v_1v_2 + i(v_2 + u_1^2v_2 - v_1^2v_2 + 2u_1u_2v_1)\\
2gf &= 2(u_1u_2 - v_1v_2 + i(u_1v_2 + u_2v_1))
\end{aligned}$$

Using $\frac{\partial\Phi}{\partial u} = \mathrm{Re}((1-g^2)f, i(1+g^2)f, 2gf)$, we compute

$$\begin{aligned}
\left\|\frac{\partial\Phi}{\partial u}\right\|^2 &= \left(u_2 - u_1^2u_2 + v_1^2u_2 + 2u_1v_1v_2\right)^2\\
&\quad + \left(-v_2 - u_1^2v_2 + v_1^2v_2 - 2u_1u_2v_1\right)^2\\
&\quad + (2u_1u_2 - 2v_1v_2)^2\\
&= u_2^2 + u_1^4u_2^2 + v_1^4u_2^2 + 4u_1^2v_1^2v_2^2 - 2u_1^2u_2^2 + 2v_1^2u_2^2 + 4u_1u_2v_1v_2\\
&\quad - 2u_1^2u_2^2v_1^2 - 4u_1u_2v_1^3v_2 - 4u_1^3u_2v_1v_2\\
&\quad + v_2^2 + u_1^4v_2^2 + v_1^4v_2^2 + 4u_1^2u_2^2v_1^2 + 2u_1^2v_2^2 - 2v_1^2v_2^2 + 4u_1u_2v_1v_2\\
&\quad - 2u_1^2v_1^2v_2^2 + 4u_1^3u_2v_1v_2 - 4u_1u_2v_1^3v_2\\
&\quad + 4u_1^2u_2^2 + 4v_1^2v_2^2 - 8u_1u_2v_1v_2\\
&= (u_2^2 + v_2^2)\left(1 + v_1^4 + u_1^4 + 2u_1^2 + 2v_1^2 + 2u_1^2v_1^2\right)\\
&= (u_2^2 + v_2^2)\left(1 + u_1^2 + v_1^2\right)^2
\end{aligned}$$

which is what we wanted to show. $\square$

Example I.1.6 (Catenoids). We want to find the surfaces C_λ^W with Weierstrass data $g(z) = z$, $\omega = \lambda z^{-2}dz$, with $\Sigma = \hat{\mathbb{C}}$ and $\{p_1, p_2\} = \{0, \infty\}$. Since we need to do the computation in the simply connected case, we lift g, ω to $\mathbb{C} = \hat{\mathbb{C}}\backslash\{\infty\}$ by exp, where we obtain $\tilde{g}(z) = g \circ \exp(z) = \exp(z)$ and $\tilde{\omega} = \omega \circ \exp$, i.e. $\tilde{w}(z) = \lambda \exp(-2z)d(\exp(z)) = \lambda\exp(-z)dz$.

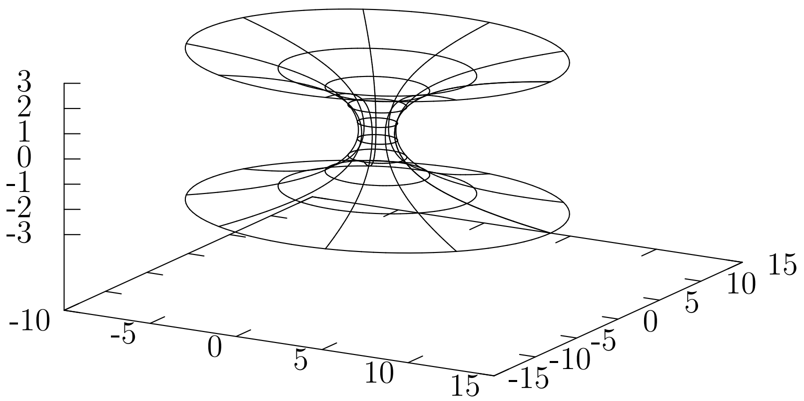

Figure I.1: A (part of a) catenoid, plotted with `GNUplot`.

Choosing $z_0 = 0$, we compute

$$\begin{aligned} C_\lambda^W(x+iy) &= \operatorname{Re}\int_0^{x+iy} \begin{pmatrix} (1-e^{2z})\lambda e^{-z}dz \\ i(1+e^{2z})\lambda e^{-z}dz \\ 2e^z\lambda e^{-z}dz \end{pmatrix} \\ &= \lambda \operatorname{Re}\int_0^{x+iy} \begin{pmatrix} (e^{-z}-e^z)dz \\ i(e^{-z}+e^z)dz \\ 2dz \end{pmatrix} \\ &= 2\lambda \begin{pmatrix} 1+\cosh x\cos y \\ -\cosh x \sin y \\ x \end{pmatrix} \end{aligned}$$

We recognize a parametrization of the catenoid, rescaled by the factor 2λ (which is why some authors prefer the factor $1/2$ in the Weierstrass representation). Moreover, since cosine and sine have a period of 2π, the catenoid is well-defined on $\mathbb{C}\backslash\{0\}$.

Further, we see that the normal vector points inward if $\lambda > 0$ and outward if $\lambda < 0$. ♢

I.2 The associate surface

Let $\Sigma, \Sigma\backslash\{p_j\}, z_0, f, g$, and Φ_W be as in the preceding section.

Following notation in [HK97], we define

Definition I.2.1. To a minimal surface Φ_W with Weierstrass data (g,ω), we define its *associate surface* $\tilde{\Phi}_W$ as the surface with Weierstrass data $(g, i\omega)$.

Observe that this definition makes sense on the universal cover $\widetilde{\Sigma\backslash\{p_j\}}$ only, since $\tilde{\Phi}_W$ is not necessarily well-defined on $\Sigma\backslash\{p_j\}$.

We try to find an explicit description of the relationship between Φ and $\tilde{\Phi}$:

Let J be the oriented 90 degree rotation on the tangent space of a surface.

Theorem I.2.2. *Let $\Phi : \Omega \to \mathbb{R}^3$ be a minimal immersion, where $\Omega \subset \mathbb{C}$ is a simply connected domain. Then*

$$d\bar{\Phi} = d\Phi \circ J \tag{I.2.1}$$

has a solution $\bar{\Phi} : \Omega \to \mathbb{R}^3$, unique up to translation. Moreover, Φ, $\bar{\Phi}$ are isometric immersions with Gauss maps $\nu = \bar{\nu}$, and $\bar{\Phi}$ is minimal.

Proof. This is [Gro01, Thm. 1.1]. □

We remark that in particular, the claim of I.2.1 being solvable for any planar simply connected domain Ω implies that it is solvable for minimal surfaces $\widetilde{\Sigma\backslash\{p_j\}} \to \mathbb{R}^3$ of the form we are considering (i.e. cases when Σ is a surface with boundary): The only case in which we cannot view $\widetilde{\Sigma\backslash\{p_j\}}$ as a subset of $\mathbb{R}^2$ is if $\Sigma = S^2$ and $\{p_j\} = \emptyset$ (however, there are no compact minimal surfaces without boundary in $\mathbb{R}^3$).

By continuity, we can extend a solution of I.2.1 from int $\left(\widetilde{\Sigma\backslash\{p_j\}}\right)$ (which we get from the Theorem) to $\partial\widetilde{\Sigma\backslash\{p_j\}}$.

Lemma I.2.3. *We have $\bar{\Phi} = \tilde{\Phi}$ (up to translations).*

Proof. The proof follows the ideas presented in [Gro01] after the statement of Theorem 1.1:

Assume that Φ is conformal, i.e. $J\frac{\partial}{\partial x} = \frac{\partial}{\partial y}$ and $J\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}$.

Then $d\Phi \circ J(\frac{\partial}{\partial x}) = \Phi_y$, and $d\Phi \circ J(\frac{\partial}{\partial y}) = -\Phi_x$.

Thus condition I.2.1 becomes

$$\bar{\Phi}_x dx + \bar{\Phi}_y dy = \Phi_y dx - \Phi_x dy.$$

Thus the condition of Theorem I.2.2 becomes the Cauchy-Riemann equations:

$$\bar{\Phi}_x = \Phi_y \qquad \bar{\Phi}_y = -\Phi_x$$

Hence the function $\Phi - i\bar{\Phi}$ is holomorphic.

By definition, the function $\Phi - i\tilde{\Phi}$ is holomorphic as well. Therefore, $\bar{\Phi}$ and $\tilde{\Phi}$ differ by a constant only. □

Lemma I.2.4. *If $\gamma : [0,1] \to \widetilde{\Sigma \backslash \{p_j\}}$ is a curve such that $\Phi_W \circ \gamma$ is contained in a plane of reflection of Φ_W, then $\tilde{\Phi}_W \circ \gamma$ is contained in a line.*

Proof. $\Phi_W \circ \gamma$ has constant conormal, since a minimal surface intersects a plane of reflection orthogonally. By Theorem I.2.2, the tangent space is rotated by a right angle when moving to $\tilde{\Phi}_W$, so the curve $\tilde{\Phi}_W \circ \gamma$ has constant tangent vector. □

Example I.2.5 (Helicoids). Let us compute the associate surfaces H_λ of C_λ^W. Working in the universal cover, we are looking for surfaces with Weierstrass data $g = \exp(z)$, $\omega = \lambda i \exp(-z) dz$.

Choosing $z_0 = 0$, we compute

$$\begin{aligned} H_\lambda(x+iy) &= \mathrm{Re} \int_0^{x+iy} \begin{pmatrix} (1-e^{2z})\lambda i e^{-z} dz \\ i(1+e^{2z})\lambda i e^{-z} dz \\ 2\lambda i dz \end{pmatrix} \\ &= \lambda \, \mathrm{Re} \int_0^{x+iy} \begin{pmatrix} i(1-e^{2z})e^{-z} dz \\ -(1+e^{2z})e^{-z} dz \\ 2idz \end{pmatrix} \\ &= \lambda \, \mathrm{Re} \int_0^{x+iy} \begin{pmatrix} i(e^{-z}-e^{z}) dz \\ -(e^{-z}+e^{z}) dz \\ 2idz \end{pmatrix} \\ &= 2\lambda \begin{pmatrix} \sinh x \sin y \\ -\sinh x \cos y \\ -y \end{pmatrix} \end{aligned}$$

We recognize the helicoid, rescaled by the factor 2λ (cf. figure I.2).
For the record, we note what the helix (choose $x = \sinh^{-1}(1)$) looks like on this surface: $y \mapsto \lambda(\sin y, -\cos y, -y)$. ◇

Definition I.2.6. We call $4\pi\lambda$ the *period* of the helicoid H_λ. So the period of a helicoid H is twice the minimal distance of two parallel lines contained in H.

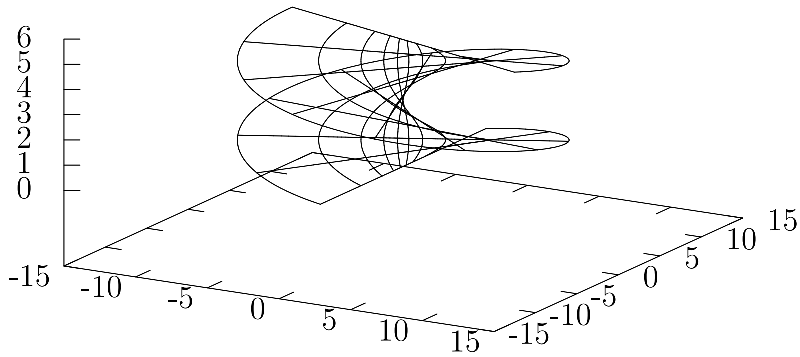

Figure I.2: A (part of a) helicoid, plotted with `GNUplot`.

By a *half-helicoid*, we denote a part of a helicoid with fixed sign of x, when parametrized as in the previous Example.

By a *helicoid-segment*, we denote a part of a helicoid with domain $[0,\infty]\times[y_1,y_2]$ (parametrized as in the Example above).

I.3 Some properties of minimal surfaces

In this section we state some facts about minimal surfaces which will be used later on.

For us, minimal surfaces with boundary are particularly important. In this context, the problem of finding minimal surfaces of prescribed boundary, also known as "Plateau Problem", is natural.

In order to state its solution in a precise way, we define the following notion: A map $\gamma : S^1 \to \Gamma$ is a *monotone parametrization* of a Jordan curve Γ if it is surjective and $\gamma^{-1}(p)$ is connected for every point $p \in \Gamma$. Let

$$\begin{aligned} X_\Gamma := \{\Psi : \bar{D} \to \mathbb{R}^n \mid\, & \Psi \text{ is piecewise } C^1, \\ & \Psi|_{\partial D} \text{ is a monotone parametrization of } \Gamma, \\ & \text{and area}(\Psi(D)) < \infty\} \end{aligned}$$

Theorem I.3.1 (Plateau Problem). *Let Γ be a Jordan curve in $\mathbb{R}^n$ such that X_Γ is not empty. Then there exists a continuous simply connected minimal surface Φ bounded by Γ, which parametrizes Γ monotonically and has minimal area among all elements of X_Γ.*

Proof. This Theorem is originally due to J. Douglas, and was proved independently by T. Rado for dimension 3. We have taken the statement from [Law80, Chapter II, Thm. 7] (where a detailed proof can be found). □

Theorem I.3.2 (Plateau solutions in $\mathbb{R}^3$ are immersed). *Every solution to the Plateau Problem in $\mathbb{R}^3$ is free of branch points, i.e., is a regularly immersed surface.*

Proof. This is [Law80, Chapter II, Thm. 9]. □

Theorem I.3.3. *A minimal surface in $\mathbb{R}^3$ is analytic in its interior.*

Proof. For minimal surfaces of the kind considered in this thesis (i.e. properly immersed surfaces with finite total curvature), this follows immediately from the Weierstrass representation, since the real part of a holomorphic function is analytic. □

We will also use that in fact all CMC-H surfaces in $\mathbb{R}^3$ or $\mathbb{H}^3$ are analytic.

Theorem I.3.4. *Let M be a three-dimensional compact manifold with piecewise smooth boundary which has the following properties:*

- *M is a compact subdomain of another smooth manifold $\bar{M}$.*
- *∂M is a two-dimensional subcomplex of $\bar{M}$ consisting of smooth two-dimensional simplices $\{H_1, \dots, H_l\}$.*
- *Each H_i is a C^2 surface in $\bar{M}$ whose mean curvature is non-negative with respect to the normal of M pointing outward.*
- *Each surface H_i is a compact subset of some smooth surfaces $\bar{H}_i$ in $\bar{M}$ where $\bar{H}_i \cap M = H_i$ and $\partial \bar{H}_i \subset \partial \bar{M}$.*

Let σ be a Jordan curve in ∂M. Then there exists a branched minimal immersion from the disk D into M with boundary σ which is smooth in the interior of D and has minimal area among all such maps. Furthermore, any branched minimal immersion of the above form must be an embedding.

Proof. This is [MY82, Thm. 1]. □

A *principal geodesic* is a geodesic which is a curvature line as well. It is not hard to see that a principal geodesic of a surface in $\mathbb{R}^3$ is a planar curve. We will discuss a generalization of this in section I.7.

Theorem I.3.5 (Schwarz Reflection Principle). *If a minimal surface (in $\mathbb{R}^3$) contains a line segment L, then it is symmetric under rotation by π about L. (If a minimal surface is bounded by a line segment L, it may be extended by rotation by π about L to a smooth minimal surface containing L in its interior.)*
If a nonplanar minimal surface contains a principal geodesic - necessarily a planar curve - then it is symmetric under reflection in the plane of that curve. (If a minimal surface meets a plane orthogonally on its boundary, the surface may be extended by reflection through the plane to a smooth minimal surface with this curve in its interior.)

Proof. This is the version of the Schwarz reflection principle which is stated and proved in [HK97, p.16f]. □

Lemma I.3.6. *A nonplanar minimal surface Φ has one of the symmetries described by the Schwarz reflection principle if and only if its associate surface has a symmetry of the other kind.*

Proof. Since $\tilde{\tilde{\Phi}} = -\Phi$, it suffices to show that if Φ has one of the symmetries from Schwarz reflection, then $\tilde{\Phi}$ has the other.

One direction of this follows from Lemma I.2.4, together with the Schwarz reflection principle.

The other direction is similar: Let $\Phi \circ \gamma$ be contained in a line (which is a line of rotational symmetry by Schwarz reflection). Then γ is a geodesic. Furthermore, the associate curve $\tilde{\Phi} \circ \gamma$ has constant conormal (by the first-order description of the associate surface given in I.2.2). So $\tilde{\Phi} \circ \gamma$ is contained in a plane, and $\tilde{\Phi}$ intersects this plane orthogonally. This implies that $\tilde{\Phi} \circ \gamma$ is a principal geodesic; hence the plane it is contained in is a symmetry plane by the Schwarz reflection principle. □

Lemma I.3.7. *An embedded end of a complete minimal surface (with compact boundary) of finite total curvature is C^1-asymptotic to a catenoid end or to a plane.*

Proof. This is [HK97, Remark 2.4.i]. □

Definition I.3.8. A sequence $(M_k)_{k\in\mathbb{N}}$ of C^1-surfaces (in some ambient Riemannian 3-manifold M) satisfies *uniform local area bounds* if there exist $C \in \mathbb{R}$ and $r > 0$ such that

$$\forall \rho < r. \forall x \in M. \forall k \in \mathbb{N}. (B_\rho(x) \cap \partial M_k = \emptyset) \Rightarrow |M_k \cap B_\rho(x)| < C$$

Theorem I.3.9. *Let N^3 be a non-compact closed manifold with boundary (whose curvature tensor is C^1-bounded) and Γ a connected non-closed curve in N, being piecewise C^1. Assume there exist compact sets $(N_n)_{n\in\mathbb{N}}$ with $N_{n-1} \subset N_n$ and $\bigcup_{n\in\mathbb{N}} N_n = N$, and closed Jordan curves $\Gamma_n \subset N_n$ with $\Gamma_k \cap N_n = \Gamma \cap N_n$ for all $k > n$. Finally let the Plateau problems for Γ_n be solvable with a sequence of minimal disks $M_n \subset N_n$ satisfying uniform local area bounds. Then there exists a minimal surface $M \subset N$ of the type of the disk which is regular in the interior and whose boundary is Γ.*

Proof. This is [Gro93, Thm. 4.4] □

Theorem I.3.10 (Maximum Principle). *Let M_1, M_2 be connected complete hypersurfaces (in $\mathbb{R}^n$) of constant mean curvatures C_1, C_2. Suppose M_1 and M_2 are tangent at an interior point x, their tangent planes are horizontal and their mean curvature vectors point in the vertical up direction. If M_1 is above M_2 in a neighborhood of x and $C_2 \geq C_1$, then $M_1 = M_2$.*

Proof. This is an immediate consequence of [Sch83, Lemma 1]; we have taken the statement from [LR85].

In the context of minimal surfaces, "the maximum principle can be stated as follows: If two minimal surfaces meet at an interior point and, locally around this point, one of the surfaces lies at one side of the other, then we conclude that both surfaces coincide around the contact point." ([PR02, p. 19]) □

Remark I.3.11 (Applying the maximum principle). We are going to apply the maximum principle in the following way: There is an unknown compact minimal surface M with known boundary.

We consider a properly immersed minimal surface C without boundary. Suppose we can place C in $\mathbb{R}^3$ in a way such that $M \cap C = \emptyset$.

Then, starting from this position, we can move C continuously (i.e. with a continuous map $F : [0,1] \to \text{Isom}(\mathbb{R}^3)$ satisfying $F(0) = id$), and if the copy $C_t = F(t)(C)$ of C satisfies $C_t \cap \partial M = \emptyset$ for all $t \in [0, \bar{t}]$, we conclude $C_{\bar{t}} \cap M = \emptyset$; otherwise, there would have to be a first point of contact at (some) $\tilde{t}$, and M would be part of $C_{\tilde{t}}$ (in which case $\partial M \cap C_{\tilde{t}} \neq \emptyset$ has to hold as well) by the maximum principle. Here, $\tilde{t} = \inf\{t \,|\, C_t \cap M \neq \emptyset\}$. Since M is compact and C is properly immersed, we have $C_{\tilde{t}} \cap M \neq \emptyset$, so that $C_{\tilde{t}}$ and M touch and the maximum principle applies.

If $\partial C \neq \emptyset$, we have to make sure that $\partial C_t \cap M = \emptyset$ for all $t \in [0, \bar{t}]$ in order to apply the maximum principle as explained before.

Apparently, we need to have some knowledge about the position of M; usually, we obtain this knowledge from the convex hull property (I.3.12).

The surface C is commonly referred to as a *barrier*. ♢

Lemma I.3.12 (Convex hull property). *A compact minimal surface in $\mathbb{R}^3$ is contained in the convex hull of its boundary.*

Proof. This can easily be inferred from the maximum principle, using planes as barriers. □

Lemma I.3.13. *Let M be a compact minimal surface in $\mathbb{R}^3$ whose boundary is a Jordan curve which projects diffeomorphically onto a convex planar curve $\Gamma \subset \{x_3 = 0\}$. Then M is a graph over the domain Ω enclosed by Γ.*

Proof. This is an elementary consequence of the maximum principle. A proof can be found in [PR02, Prop. 1.1]. □

From the Plateau problem and the maximum principle, we can deduce the following:

Corollary I.3.14. *Let Γ be a Jordan curve in $\mathbb{R}^3$ which projects diffeomorphically onto a convex planar curve in the xy-plane. Then there exists a compact minimal disk M bounded by Γ. Furthermore, M is unique, area-minimizing and a graph over the xy-plane.* □

At this point we wish to point out that there is a version of the maximum principle "at infinity", which is an important open problem for the global theory of minimal surfaces:

Conjecture I.3.15 (Maximum Principle at Infinity, MPI). *Suppose M_1 and M_2 are two disjoint properly immersed minimal surfaces in a complete flat 3-manifold. Then*

$$d(M_1, M_2) = \min\{d(\partial M_1, M_2), d(M_1, \partial M_2)\}.$$

In case ∂M_1 and ∂M_2 are empty, M_1 and M_2 are flat totally geodesic constant distance hypersurfaces. □

The case of surfaces without boundary has already been shown, and is known as the *half-space Theorem*.

We have taken the formulation of the conjecture from [Mee02, sec. 1], where the interested reader can also find some remarks and references about progress towards proving the maximum principle at infinity.

Remark I.3.16 (Applying the maximum principle at infinity). If the maximum principle at infinity holds, it can be applied in almost the same way as the maximum principle:

The assumption of one of the surfaces being compact can be dropped. Let $C_t := F(t)C$, where F is a path in $\mathrm{Isom}(\mathbb{R}^3)$ starting at id as above. If $C_0 \cap M = \emptyset$ and we can make sure that $d(\partial C_t, M)$ and $d(C_t, \partial M)$ are bounded below for $t \in [0, \bar{t}]$, we can conclude $C_{\bar{t}} \cap M = \emptyset$. ♢

There are some related results for non-compact minimal surfaces:

Theorem I.3.17. *Suppose $M \subset \{x_3 \geq 0\}$ is a properly immersed minimal surface in $\mathbb{R}^3$ with nonempty, possibly noncompact, boundary.*

If $x_3(\partial M) \geq \delta$, then $x_3(M) \geq \delta$.

Proof. This is [MR93, Lemma 2.1]. □

This means that the MPI holds when planes are used as barriers and the comparison surface has nonempty boundary.

Theorem I.3.18. *Any connected properly immersed minimal surface in a convex slab wedge lies in the convex hull of its boundary.*[1]

Proof. This is [LM98, Thm. 5]. □

Remark I.3.19. Observe that for any non-planar minimal surface M satisfying the convex hull property we have: M touches its convex hull in boundary points only. ♢

I.4 Hyperbolic Space

In this section, we give a very brief introduction to (three-dimensional) hyperbolic space and its models which we will use.

Except for the hermitian model, the models generalize immediately to dimensions other than 3.

For more detailed introductions to hyperbolic geometry, we recommend [BP92] and [Rat94].

This section comprises material from [Rat94], [Ross01] and [Bry87].

I.4.1 The Hyperboloid Model

We consider the Lorentz 4-space denoted by $L(4)$, which is $\mathbb{R}^4$ with the following Riemannian metric: Let a point be parametrized by (x_0, x_1, x_2, x_3). Then we consider the Riemannian metric on $\mathbb{R}^4$ given by $-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$; that is to say we consider the bilinear function

$$\langle (x_0, x_1, x_2, x_3), (y_0, y_1, y_2, y_3) \rangle_L := -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3$$

In this space, we define:

$$\mathbb{H}^3 := \{ v = (v_0, v_1, v_2, v_3) \in L(4) \,|\, v_0 > 0 \wedge \langle v, v \rangle_L = -1 \}$$

[1] The definition of a convex slab wedge can be found at the beginning of section II.1, since we will need the notion there.

So $\mathbb{H}^3$ is one sheet of a hyperboloid, and we call $\mathbb{H}^3$ hyperbolic 3-space.

It can easily be seen that the bilinear function defined above is positive definite when restricted to the tangent space of $\mathbb{H}^3$.

Hence, $\mathbb{H}^3$ is a Riemannian manifold. A geodesic is the intersection of $\mathbb{H}^3$ with a (2-)plane in $L(4)$ through the origin.

The metric on $\mathbb{H}^3$ induced by the Riemannian metric on its tangent vectors can be computed to be

$$\cosh d_L(x, y) = -\langle x, y\rangle_L .$$

I.4.2 The Poincaré Disk Model

Using stereographic projection of the hyperboloid introduced above to the unit disk in $\{x_0 = 0\}$ from the point $(-1, 0, 0, 0)$ yields the Poincaré disk model. In this model, we can see the *asymptotic boundary* $\partial\mathbb{H}^3$ of hyperbolic space as the unit sphere. The geodesics in this model are segments of circles intersecting $\partial\mathbb{H}^3$ orthogonally.

It can be seen that the projection is conformal (i.e. preserves angles), and that the metric induced on the unit disk by this projection is given by

$$\cosh d_B(x, y) = 1 + \frac{2\|x - y\|^2}{(1 - \|x\|^2)(1 - \|y\|^2)}.$$

I.4.3 The Upper Half-Space Model

To obtain the upper half-space model from the Poincaré disk model, we apply the map $\eta := \rho\sigma$ to the unit ball; here, ρ is reflection in the hyperplane $\{x_3 = 0\}$ and σ is inversion in the sphere of radius $\sqrt{2}$ around the point $(0, 0, 1)$.
The image of the unit ball under η is the upper half-space $U = \{x_3 > 0\}$, and $\partial\mathbb{H}^3 = \{x_3 = 0\} \cup \{\infty\}$ in this model. We compose the inversion and the reflection for the transition map η to be orientation-preserving.

The geodesics are semi-circles intersecting $\{x_3 = 0\}$ orthogonally, or rays parallel to the x_3-axis starting in the hyperplane $\{x_3 = 0\}$.

Furthermore, η is conformal since inversion in a sphere is, and the induced metric on the upper half-space can be computed to be

$$\cosh d_U(x, y) = 1 + \frac{\|x - y\|^2}{2x_3y_3}.$$

I.4.4 The Hermitian Model

We consider the bijection from $L(4)$ to the space of 2×2 hermitian symmetric matrices by mapping the point $v = (v_0, v_1, v_2, v_3) \in L(4)$ to the matrix

$$\begin{pmatrix} v_0 + v_3 & v_1 + iv_2 \\ v_1 - iv_2 & v_0 - v_3 \end{pmatrix}$$

Using this identification, we have

$$\det v = v_0^2 - v_3^2 - v_1^2 - v_2^2 = - \langle v, v \rangle_L .$$

Thus, another model of hyperbolic space is given by considering positive definite Hermitian symmetric matrices of determinant 1.

Observe that every matrix A in $SL(2, \mathbb{C})$ gives rise to an element of hyperbolic space via the formula AA^*; every element of hyperbolic space can be obtained in this way (but not uniquely).

I.5 Bryant Representation of CMC-1 surfaces in $\mathbb{H}^3$

Definition I.5.1. We define a *Bryant-surface* to be an immersed CMC-1 surface in $\mathbb{H}^3$ (This notation was introduced by Rosenberg, cf. [CHR01], [Ros02]).

Definition I.5.2. For a Bryant surface M, we define its *asymptotic boundary* to be $\bar{M} \cap \partial\mathbb{H}^3$, where $\bar{M}$ is the closure of M when viewed as a subset of the unit ball (i.e. in the Poincaré disk model of hyperbolic space) with the Euclidean topology.

For CMC-1 surfaces in $\mathbb{H}^3$, Bryant (in [Bry87]) has found a representation similar to the Weierstrass representation. We state the version from [Ross01, Thm. 3].

Theorem I.5.3. *Let $\Sigma, \Sigma\backslash\{p_j\}, z, z_0, f,$ and g be as in the Weierstrass representation (Thm. I.1.1).*
Choose the holomorphic immersion $F : \widetilde{\Sigma\backslash\{p_j\}} \to SL(2, \mathbb{C})$ so that $F(z_0)$ is the identity matrix and F satisfies

$$dF = F \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} f dz.$$

Then $\Phi_B : \widetilde{\Sigma\backslash\{p_j\}} \to \mathbb{H}^3$ defined by

$$\Phi_B = F \cdot F^*$$

is a conformal CMC-1 immersion in the Hermitian model of $\mathbb{H}^3$. Furthermore, any CMC-1 surface with finite total curvature in $\mathbb{H}^3$ can be represented this way. □

Definition I.5.4. Let $\Sigma, \Sigma\backslash\{p_j\}, z, z_0, f,$ and g be as in the Weierstrass representation (Thm. I.1.1). Then we define the surface Φ_B from Theorem I.5.3 to be the *(Bryant) cousin* of the minimal surface Φ_W from Theorem I.1.1. For a Bryant surface in $\mathbb{H}^3$, we will similarly speak of its *minimal cousin* in $\mathbb{R}^3$.

Remark I.5.5. Similar to remark I.1.4, the Bryant representation and the Bryant cousin relation work for all simply connected Bryant surfaces. ◇

Lemma I.5.6. *Any minimal surface Φ_W in $\mathbb{R}^3$ is isometric to its Bryant cousin Φ.*

Proof. According to Lemma I.1.5, it suffices to show that the Riemannian metric on Φ is given by $(1 + g\bar{g})^2 f\bar{f}$. We check this well-known fact by calculation:

The metric in the Hermitian model of $\mathbb{H}^3$ is given by $\langle v, v\rangle_L = -\det v$ (as we have seen in section I.4.3). Hence, we compute $\det(d\Phi/dz)$:

$$\Phi = F \cdot F^*$$

and so

$$\begin{aligned}(d\Phi/dz) &= (dF/dz) \cdot F^* + F \cdot (dF^*/dz) \\ &= F \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} f F^* + F \cdot \left(F \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} f \right)^* \\ &= F \begin{pmatrix} gf + \bar{g}\bar{f} & \bar{f} - g^2 f \\ f - \bar{g}^2\bar{f} & -gf - \bar{g}\bar{f} \end{pmatrix} F^*\end{aligned}$$

Since $F \in SL(2, \mathbb{C})$ by assumption, we have

$$\begin{aligned}\det(d\Phi/dz) &= (gf + \bar{g}\bar{f})(-gf - \bar{g}\bar{f}) - (\bar{f} - g^2 f)(f - \bar{g}^2\bar{f}) \\ &= -g^2 f^2 - 2g\bar{g}f\bar{f} - \bar{g}^2\bar{f}^2 \\ &\quad - f\bar{f} + \bar{g}^2\bar{f}^2 + g^2 f^2 - g^2\bar{g}^2 f\bar{f} \\ &= -f\bar{f}(1 + g\bar{g})^2\end{aligned}$$

which is what we wanted to show. □

Following conventions in [Bry87] and [ET01], we define:

Definition I.5.7. For $\lambda \neq 0$, we denote the cousin of C_λ^W from example I.1.6 by $C_\lambda : \mathbb{C} \to \mathbb{H}^3$. If $\lambda > -\frac{1}{4}$, we call C_λ a *catenoid cousin*.

The catenoid cousins are surfaces of revolution (and in fact, the only Bryant surfaces of revolution), are embedded if and only if $\lambda > 0$, and have a circle of self-intersection otherwise (see [Bry87], [UY93], [Ros02, sec. 11]).

It is an interesting fact that the surfaces C_λ with $\lambda \leq -1/4$ are not surfaces of revolution in hyperbolic space (cf. example I.7.8).

Example I.5.8 (Catenoid Cousins). The catenoid cousins are explicitly determined in [Ros02, sec. 11]. Rosenberg calculates the formulas in the upper half space model of $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_{>0} = \{(u+iv, w) \mid u, v \in \mathbb{R}, w > 0\}$, and the catenoid cousins are parametrized as maps $\mathbb{C} \to \mathbb{H}^3$, the two ends being located at $0, \infty$ (this is [Ros02, p.86], where we use that $\alpha = \frac{1}{2}\sqrt{1+4\lambda} = \sqrt{\lambda + 1/4}$ in Rosenberg's notation):

$$C_\lambda : \begin{cases} u + iv = \dfrac{-\lambda(e^x + e^{-x})e^{x\sqrt{1+4\lambda}}}{\left(\frac{1}{2} + \lambda - \frac{1}{2}\sqrt{1+4\lambda}\right) e^{-x} + \left(\frac{1}{2} + \lambda + \frac{1}{2}\sqrt{1+4\lambda}\right) e^{x}} e^{iy\sqrt{1+4\lambda}} \\ w = \dfrac{\sqrt{1+4\lambda}e^{x\sqrt{1+4\lambda}}}{\left(\frac{1}{2} + \lambda - \frac{1}{2}\sqrt{1+4\lambda}\right) e^{-x} + \left(\frac{1}{2} + \lambda + \frac{1}{2}\sqrt{1+4\lambda}\right) e^{x}} \end{cases} \tag{I.5.1}$$

Observe that C_λ is a surface of revolution in hyperbolic space: The lines $x = \text{const}$ are mapped to circles centered at $(0, 0, w(x))$.

Note that all catenoids constructed in example I.1.6 are 2π-periodic in the y-variable. This is not true for the catenoid cousins: They are also periodic in the y-variable, but the period is $\Delta y = \frac{2\pi}{\sqrt{1+4\lambda}}$.

For later use, we define the function $\tilde{\varphi}(\lambda) := \frac{\pi}{\sqrt{1+4\lambda}}$, which is a bijective map $(-\frac{1}{4}, \infty)\backslash\{0\} \to (0, \infty)\backslash\{\pi\}$. $\diamondsuit$

I.6 Lawson's Correspondence

Let $M^n(c)$ denote the n-dimensional *model space* of curvature $c \in \mathbb{R}$; i.e. $M^n(c)$ is the simply connected complete n-dimensional Riemannian manifold of constant sectional curvature c.

Lawson's correspondence relates CMC-H surfaces in 3-dimensional model spaces of different curvature, and was discovered in [Law70]. Our description is a blend of the presentations in [Ros02, pp. 69f] and [Ross01, pp. 675f].

Consider a simply connected Riemannian surface (M, ds^2), together with a smooth field of symmetric transformations $S : T_xM \to T_xM$. Let K_{ds^2} denote the (intrinsic) Gauss curvature of (M, ds^2), and let ∇ denote the Riemannian connection of $M^3(c)$. Then the integrability conditions are the *Gauss equation*

$$K_{ds^2} = \det S + c$$

and the *Codazzi equation*

$$\langle S([X, Y]), Z\rangle = \langle \nabla_X S(Y), Z\rangle - \langle \nabla_Y S(X), Z\rangle$$

which, if they hold (for all X, Y, Z), assure that there exists an isometric immersion of M in $M^3(c)$ with shape operator S.

If (M, ds^2, S) satisfies these equations and $\operatorname{trace} S \equiv H$, then M can be realized as CMC-H surface in $M^3(c)$.

Now choose $d \in \mathbb{R}$ and let

$$\tilde{S} := S + d \cdot id, \qquad \tilde{c} = c - 2dH - d^2.$$

It is immediate to check that the Codazzi equation is still satisfied, and the Gauss equation is satisfied in $M^3(\tilde{c})$.

Thus, we have obtained a CMC-$\tilde{H}$ surface in $M^3(\tilde{c})$, where $\tilde{H} = H + d$.

The case we are interested in is $H = c = 0$. We observe that we can choose $d = \pm 1$ to obtain CMC-1 surfaces in $\mathbb{H}^3$.

Similarly, every CMC-1 surface in $\mathbb{H}^3$ has a *minimal Lawson cousin.*

Lemma I.6.1. *The Bryant representation is a special case of Lawson's correspondence (and $\tilde{S} = S + id$, where S is the shape operator of the minimal surface in $\mathbb{R}^3$ and $\tilde{S}$ is the shape operator of the CMC-1 surface in $\mathbb{H}^3$).*

Proof. This is a special case of [UY92, Thm. 3.1].

Note that we have already proved one part in Lemma I.5.6. Showing the relation of the second fundamental forms is a similar calculation (involving more facts about the Weierstrass representation than we wanted to present here). □

Remark I.6.2. We make the following important observation at this point: While regarding surfaces in $\mathbb{R}^3$ which differ by an orientation-preserving isometry as "essentially equal", we have to consider surfaces as different if they are isometric by an orientation-reversing isometry:

In the case of catenoids, the orientation-reversing isometry $x \mapsto -x$ even leaves the surface invariant. However, the corresponding Bryant cousins C_λ and $C_{-\lambda}$ look very different (cf. examples I.1.6, I.2.5, I.5.8, I.7.8).

The reason is the following: Consider a minimal surface M with shape operator S. Let $M' := (-1) \cdot M$; then the shape operator S' of M' satisfies $S' = -S$.

Now the Lawson cousin of M with shape operator $S - id$ and the Lawson cousin of M' with shape operator $S' + id = -S + id = (-1) \cdot (S - id)$ differ by orientation only.

Hence, the difference when considering Bryant cousins of M and M' is essentially the same as choosing whether to add or subtract id to the shape operator S (in Lawson's correspondence). ◇

I.7 Transformation of Symmetries

We need some information about the relationship between symmetries of a Bryant surface Φ and symmetries of its minimal cousin.

We start by citing a Lemma from [ET01]:

Lemma I.7.1 ([ET01, 3.1]). *Let M be a surface in $\mathbb{H}^3$. Let $\alpha : (-1, 1) \to M$ be a principal geodesic of M. Then α is a planar curve; that is, α stays on a geodesic plane in $\mathbb{H}^3$. Furthermore, M is orthogonal to this geodesic plane along α.* □

Also, we cite the following Theorem from [ET01]:

Proposition I.7.2 ([ET01, 3.2]). *Let $X : U \to \mathbb{R}^3$ be a nonplanar minimal immersion, where U is a simply connected planar domain.*
Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be a reflection in a plane such that $R(X(U)) = X(U)$. Then the associated constant mean curvature 1 surface in $\mathbb{H}^3$ is invariant under reflection in a hyperbolic plane of $\mathbb{H}^3$.

However, their proof contains the erroneous statement that a curve of intersection between the symmetry plane and $X(U)$ is a principal geodesic, which is not true in general.

Their proof shows:

Lemma I.7.3. *A principal geodesic of a Bryant surface is a curve of planar symmetry, and a principal geodesic of a minimal surface is mapped to a principal geodesic by the Bryant cousin relation.* □

We can sharpen this statement as follows:

Lemma I.7.4. *A curve $\gamma : [0, 1] \to \widetilde{\Sigma \backslash \{p_j\}}$ is a principal geodesic in the minimal surface Φ_W if and only if γ is a principal geodesic in the Bryant surface Φ_B.*

Proof. Since Φ_W and Φ_B are isometric, being a geodesic is invariant under the cousin construction. Since the shape operator S' of Φ_B is $S + id$ (where S is the shape operator of Φ_W) by Lawson's correspondence, the same holds for being a curvature line. □

For the statement of the following Proposition, we call a surface Φ *well-reflectable* in a plane P if a component of $\Phi^{-1}(P)$ is a principal geodesic.

Proposition I.7.5. *A minimal surface is well-reflectable in a plane if and only if its Bryant cousin is.* □

We mention that similar results do not hold for rotational symmetries, but we refer the reader to [ET01] for details.

Remark I.7.6. Observe that if a curve of intersection between a nonplanar Bryant surface and a symmetry plane has precisely one preimage, then the curve of intersection is necessarily a principal geodesic. ♢

Lemma I.7.7. *Let $\Phi : \Sigma\backslash\{p_j\} \to X$ be a minimal surface in $\mathbb{R}^3$ or a Bryant surface, which is properly immersed. Let $\gamma : (0,1) \to \Sigma\backslash\{p_j\}$ be a principal geodesic, and let $\tilde{\gamma}$ be the maximal geodesic which γ is part of. Then $\tilde{\gamma}$ is a principal geodesic.*

Proof. We know that Φ is invariant under reflection in a plane P which contains $\Phi \circ \gamma$. By uniqueness of geodesics, $\Phi \circ \tilde{\gamma}$ is contained in P.

Since γ is a principal geodesic, Φ intersects P orthogonally along γ. Because Φ is analytic, it follows that Φ intersects P orthogonally along $\tilde{\gamma}$, which finishes the proof. □

Example I.7.8. As we have mentioned, the surfaces C_λ for $\lambda < -1/4$ are not rotational. However, they are subject to a family of symmetry planes by Proposition I.7.5. It is interesting to check what these symmetry planes look like.

The formulas are calculated in [Ros02, p.87]. We let $(i\beta)^2 = 1/4 + \lambda$, with $\beta > 0$. Observe that there is a small typo in Rosenberg's formula for w, which we corrected in the formulas we give here (the notation is as in example I.5.8):

$$C_\lambda : \begin{cases} u + iv = \dfrac{-i\left(\left(\frac{1}{2} + i\beta\right)^2 e^{-x} + \left(\frac{1}{2} - i\beta\right)^2 e^{x}\right) e^{2i\beta x}}{\left(\frac{1}{4} + \beta^2\right)(e^x + e^{-x})} e^{-2\beta y} \\ w = \dfrac{2\beta}{\left(\frac{1}{4} + \beta^2\right)(e^x + e^{-x})} e^{-2\beta y} \end{cases}$$

We observe that for any point $C_\lambda(x_0 + iy_0)$ in $C_\lambda(\mathbb{C})$, the ray starting at 0 through $C_\lambda(x + iy)$ is contained in the surface (note that these rays are parametrized by lines of the form $\{x = x_0\}$). Hence, C_λ is invariant by reflection in any hemisphere centered at the origin.

We know that the lines $\{y = \text{const}\}$ are lines of planar symmetry for the catenoids, which are intersected orthogonally by the lines $\{x = \text{const}\}$. Since C_λ^W and C_λ are isometric, the same has to hold for the lines $\{y = \text{const}\}$ on C_λ. Therefore, such a line has to be contained in a hemisphere centered at the origin. ♢

I.8 Asymptotic behavior of Bryant surfaces

Recently, the following Theorem has been established:

Theorem I.8.1. *Let E be a properly embedded CMC-1 annular end in $\mathbb{H}^3$. Then E is (exponentially) asymptotic to a catenoid cousin end or to a horosphere end.*

Proof. This is [CHR01, Thm. 10]. The fact that the asymptotics is exponential follows from work in [ET01]. □

Theorem I.8.2. *Let M be an embedded Bryant surface. If M is not a horosphere, every annular end of M is asymptotic to a catenoid cousin.*

Proof. This is [CHR01, Thm. 12]. □

Lemma I.8.3. *The mean curvature normal of an embedded Bryant surface points inwards.*

Proof. This follows immediately from [RR98, Thm. 1], which says that for a surface disjoint from a horosphere, it is either a horosphere, or its mean curvature normal does not point to this horosphere. Since we know that every end corresponds to one point on $\partial\mathbb{H}^3$, we can find a suitable horosphere. □

We use the definition of Alexandrov-embeddedness as stated in [GKS01]:

Definition I.8.4. A CMC surface M of finite topology is *Alexandrov-embedded* if M is properly immersed, if each end of M is embedded, and if there exists a compact three-manifold W with boundary $\partial W =: \Sigma$ and a proper immersion $F : W\backslash\{q_1, \dots, q_k\} \to \mathbb{R}^3$ whose boundary restriction $f : \Sigma\backslash\{q_1, \dots, q_k\} \to \mathbb{R}^3$ parametrizes M. Moreover, we require that the mean-curvature normal of M points into W. (See also [Ale58], [KKS89])

Remark I.8.5. We believe that the Theorem and the Lemma above generalize to the class of Alexandrov-embedded surfaces. ♢

Proposition I.8.6. *Let $\tilde{E}$ be a properly embedded CMC-1 annular end in $\mathbb{H}^3$. Then $\tilde{E}$ is asymptotic to a catenoid cousin end C_λ if and only if its minimal cousin E is C^1-asymptotic to C_λ^W.*

Proof. This follows immediately from the explicit representation of minimal surfaces and their Bryant cousins (for the minimal surfaces, we use the assumption of C^1-asymptotics to conclude that the Gauss maps g, g' converge to the same value). □

I.9 The conjugate surface

Definition I.9.1. Given a Bryant surface $\Phi : \widetilde{\Sigma \setminus \{p_j\}} \to \mathbb{H}^3$, we define its *conjugate surface* $\Phi^c : \widetilde{\Sigma \setminus \{p_j\}} \to \mathbb{R}^3$ as the associate surface of Φ's minimal cousin.

Proposition I.9.2. *Let $\Phi : \widetilde{\Sigma \setminus \{p_j\}} \to \mathbb{H}^3$ be a Bryant surface. The image of a principal geodesic of Φ under Φ^c is contained in a line. Conversely, every line segment contained in Φ^c is the image of a curve in $\mathbb{H}^3$ contained in a hyperbolic plane of reflection.*

Proof. This follows from Lemma I.7.4 and Lemma I.3.6. □

Proposition I.9.3. *Let E be a properly embedded CMC-1 annular end in $\mathbb{H}^3$. Then E is asymptotic to a catenoid cousin end C_λ if and only if its minimal conjugate E^c is C^1-asymptotic to the helicoid H_λ.*

Proof. This is an immediate consequence of Proposition I.8.6 and the associate construction. □

Chapter II

Minimal surfaces with an end bounded by rays

In this chapter we examine minimal surfaces with ends bounded by rays and construct some examples relevant for the chapter on Trinoids.

II.1 Notation and Barriers

First, we set up notation, construct barriers, and examine some special cases of sequences of minimal disks:

We are particularly interested in ends bounded by rays and contained in a wedge of a slab of convex angle (which we assume to be closed and call *convex slab wedge (of angle $\vartheta < \pi$)* in the following).

Definition II.1.1. A properly immersed minimal surface

$$E : \bar{D}\backslash\{x_0 \in \partial D\} \to \mathbb{R}^3$$

is called a *minimal end bounded by rays* if there is a neighborhood U of x_0 such that the two components of $U \cap \partial D\backslash\{x_0\}$ are mapped to rays.

A *ray-ended curve* is a piecewise smooth map $(0,1) \to \mathbb{R}^3$ such that a neighborhood of 0 and a neighborhood of 1 are mapped to monotonically parametrized rays.

We start with an elementary observation:

Lemma II.1.2. *Let $E : \bar{D}\backslash\{p_j \,|\, 1 \leq j \leq k\} \to \mathbb{R}^3$ (where $\{p_j\} \subset \partial D$) be a nonplanar properly immersed minimal disk contained in a convex slab wedge W. If P is a plane such that $P \cap W$ is compact and P intersects ∂E in two points, then $E^{-1}(P)$ is a simple curve.*

Proof. Consider $C := \{x \in \bar{D} : E(x) \in P\}$. Observe that C is closed and has empty interior (as a subset of $\mathbb{R}^2$; otherwise, $E(D) \subset P$ by analyticity; a contradiction). Clearly, we have $C \cap \partial D = \{y_1, y_2\}$ for two distinct points y_1, y_2. Since E is minimal, we know that $C \cap D$ consists of 1-dimensional manifolds without boundary (we do not yet know that they do not intersect one another).

However, no component K of $D \backslash C$ can have the property that $K \cup C$ is compact (otherwise we obtain a contradiction as before). Therefore C consists of exactly one arc joining y_1, y_2 (without selfintersections). □

For a ray l, we denote the line it is contained in by $\tilde{l}$, and we call two rays l_1, l_2 *non-parallel* if $\tilde{l}_1, \tilde{l}_2$ are not parallel.

Definition II.1.3. For two non-parallel rays l_1, l_2 which are not contained in a plane, we denote by $\tilde{H}_{l_1,l_2}$ the *unique* helicoid with the following properties:

- $\tilde{H}_{l_1,l_2}$ contains l_1 and l_2.
- The end of l_1 is mapped to the end of l_2 when l_1 is moved to $\tilde{l}_2$ in $\tilde{H}_{l_1,l_2}$.
- The rotating angle of l_1 when moving to $\tilde{l}_2$ in $\tilde{H}_{l_1,l_2}$ is less than π.

By $\bar{H}_{l_1,l_2}$, we denote the half-helicoid (recall definition I.2.6) of $\tilde{H}_{l_1,l_2}$ which contains the ends of l_1 and l_2.

By H_{l_1,l_2}, we denote the segment of $\bar{H}_{l_1,l_2}$ between l_1 and l_2.

For two rays l_1, l_2 which are contained in a plane P, define $\tilde{H}_{l_1,l_2} := P$ and H_{l_1,l_2} to be the sector of P which is spanned by the ends of l_1, l_2 and is of convex angle.

Definition II.1.4. Let two non-parallel rays l_1, l_2 be given such that $l_1 \subset \{z = h\}$ and $l_2 \subset \{z = -h\}$.

For $\varepsilon > 0$, we define $S_\varepsilon := \{-\varepsilon - h \leq z \leq h + \varepsilon\}$ and W_ε to be the smallest wedge of the slab S_ε which contains H_{l_1,l_2}.

Lemma II.1.5. *Let Γ be a connected non-closed simple curve in $\mathbb{R}^3$ contained in the slab $S := \{-h \leq z \leq h\}$ for some $h > 0$.*

Assume that Γ projects diffeomorphically to a curve Γ_{pl} in the xy-plane.

If a component C of $\mathbb{R}^2 \backslash \Gamma_{pl}$ is convex, then there exists a minimal disk M which is proper and regular in the interior, whose boundary is Γ, and which is contained in S.

Furthermore, M is graph over the xy-plane.

Proof. Observe that we can easily obtain a sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of Jordan curves in S satisfying the assumptions of Corollary I.3.14 and Theorem I.3.9.

For $n \in \mathbb{N}$, let M_n be the area-minimizing surface bounded by Γ_n obtained from I.3.14.

In order to apply Theorem I.3.9, we need only show that the sequence $(M_n)_{n\in\mathbb{N}}$ satisfies uniform local area bounds.

For $p \in \mathbb{R}^3$, let $C(p)$ be the solid cylinder of radius 1 about the vertical line through p.

Fix some $n \in N$. We want to show that $|M_n \cap C(p)|$ is bounded (independently from n), which clearly implies uniform local area bounds.

First, assume $C(p)\cap\partial M_n = \emptyset$. Then either $C(p)\cap M_n = \emptyset$, or $M_n\cap\partial C(p)$ is a Jordan curve A (since M_n is graph over the xy-plane).

Since M_n minimizes area, the area bounded by A is less than $4\pi h + \pi$ (since we easily obtain a piecewise differentiable comparison surface bounded by A of (at most) that area).

If $\partial M_n \cap C(p) \neq \emptyset$, let C_n be the domain enclosed by the projection of Γ_n to the xy-plane. Then $(M_n \cap C(p)) \subset (C(p) \cap (C_n \times \mathbb{R})) =: D$. As above, $M_n \cap \partial D$ is a Jordan curve A, and the area enclosed by A is bounded by $4\pi h + \pi$ as before, since the surface area of the part of D contained in the slab $\{-h \leq z \leq h\}$ is bounded by the surface area of $C(p)$ in S (since C_n is convex).

Since n was arbitrary, we can apply Theorem I.3.9 to obtain a minimal surface M bounded by Γ. Then M is graph over the xy-plane and contained in S since all surfaces M_n have these properties. □

Next, we will construct barriers B_b $(b > 0)$. The surfaces we discuss here are inspired by the surfaces developed in [CHM89, sect. 3.2].

Let Γ_b $(b > 0)$ be the piecewise smooth curve comprising the following segments:

$$\{(-x, 0, 0)\,|\,x \geq b + 1\}, \{(x, 0, 0)\,|\,x \geq b + 1\},$$
$$\{(-b - 1 + x/b, 0, x)\,|\,0 \leq x \leq b\}, \{(b + 1 - x/b, 0, x)\,|\,0 \leq x \leq b\},$$
$$\{(x, 0, b)\,|\, -b \leq x \leq b\}$$

Corollary II.1.6. *For every $b > 0$, there exists a minimal disk bounded by Γ_b contained in a vertical half-slab, which is proper and regular in the interior.*

Proof. The existence follows immediately from Lemma II.1.5.
For given $b > 0$, we use particular curves in the construction of the previous proof: For any $t > b + 1$, we obtain a Jordan curve $\Gamma_{b,t}$ as follows:

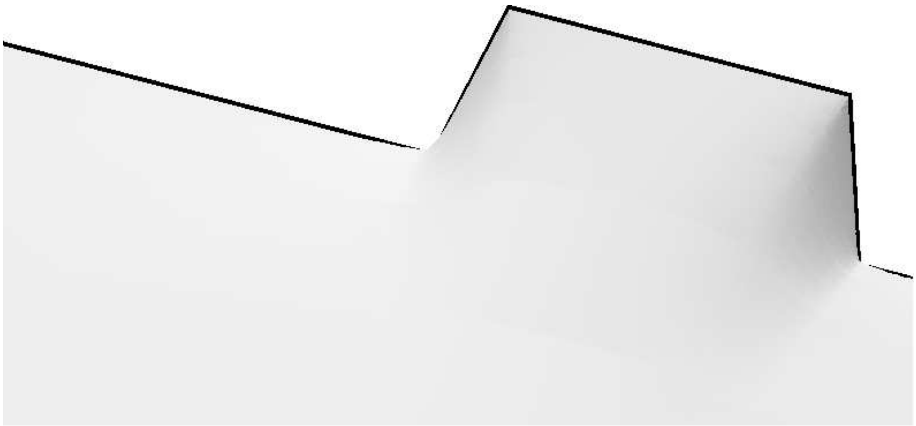

Figure II.1: A surface B_b drawn with `surface evolver`.

Connect the points $(\pm t, 0, 0)$ to the point $(0, t, 0)$ by line segments and denote the Jordan curve obtained by adding the compact part of Γ_b between $(\pm t, 0, 0)$ by $\Gamma_{b,t}$.

Now we define B_b to be the minimal surface which we obtain from Lemma II.1.5 with the curves $(\Gamma_{b,n})_{b+1<n\in\mathbb{N}}$.

Then B_b is contained in $\{y \geq 0\}$ since every $\Gamma_{b,t}$ is. □

Lemma II.1.7. *For every $b > 0$, the surface B_b is embedded, has total curvature 2π, and is C^1-asymptotic to the halfplane $\{x = z = 0, y \geq 0\}$.*

Proof. We know from II.1.5 that B_b is graph over the xy-plane; hence, it is embedded.

Let $B_{b,t}$ be the solution of the Plateau problem for $\Gamma_{b,t}$. Every curve $\Gamma_{b,t}$ has total curvature 4π, so every surface $B_{b,t}$ has total curvature 2π by Gauss-Bonnet. Thus the total curvature of B_b is 2π as well.

Extending B_b along the x-axis by the Schwarz reflection principle, we obtain a properly embedded minimal annulus of finite total curvature contained in a slab. It follows that this annular surface converges to a horizontal plane at infinity (by Lemma I.3.7), and that plane must be the xy-plane. □

Proposition II.1.8 (Uniqueness of B_b). *Let M be a properly immersed minimal disk bounded by Γ_b contained in a vertical slab and the half-space $\{y \geq 0\}$.*

If M is asymptotic to the half-plane $\{x = z = 0, y \geq 0\}$, then $M = B_b$.

Proof. Choose $\varepsilon > 0$. Then there exists $R_\varepsilon > 0$ such that

$$M \cap \{(x, y, z) \,|\, x^2 + y^2 \geq R_\varepsilon\} \subset \{-\varepsilon \leq z \leq \varepsilon\} \supset B_b \cap \{(x, y, z) \,|\, x^2 + y^2 \geq R_\varepsilon\}.$$

Consider the compact parts $M^\varepsilon := M \cap \{(x, y, z) \,|\, x^2 + y^2 \leq R_\varepsilon\}$ and $B_b^\varepsilon := B \cap \{(x, y, z) \,|\, x^2 + y^2 \leq R_\varepsilon\}$. Sliding B_b^ε up and down and applying the maximum principle, we conclude that M_ε is contained between $B_b + (0, 0, 2\varepsilon)$ and $B_b - (0, 0, 2\varepsilon)$. Thus, M is contained between these two translated versions of B_b. Since ε was arbitrary, this finishes the proof. □

II.2 Classifying Minimal Ends bounded by Rays

First, we show that some minimal surfaces can be used as barriers for minimal surfaces contained in a wedge of a half-space.

Later we use that to show that minimal ends bounded by rays l_1, l_2 which are contained in a convex slab wedge are asymptotic to H_{l_1,l_2} (in distance).

Definition II.2.1. Consider a closed sector S in the xy-plane. Then we call

$$W_S := \{(x, y, z) \in \mathbb{R}^3 \,|\, z \geq 0 \wedge (x, y, 0) \in S\}$$

a *wedge of the half-space* $\{z \geq 0\}$. A *wedge of a half-space* is the image of a wedge of the half-space $\{z \geq 0\}$ under an isometry of $\mathbb{R}^3$. We call a wedge of a half-space *convex* if the sector S in its determining plane is strictly convex (i.e. the opening angle of S is less than π).

Definition II.2.2. Let B be a properly immersed minimal surface (with boundary), and let W_S be a convex wedge of the half-space $\{z \geq 0\}$ as above. We say that B is in *good position* with respect to W_S if

- $\partial B \cap W_S = \emptyset$
- Let P_B be the orthogonal projection of B to the xy-plane. We require that $S \backslash P_B$ is bounded.
- There exists a line l such that the following holds (where R_ϑ denotes rotation of angle ϑ about l):
  - $R_{\pi/2}B \subset \{z \leq 0\}$ or $R_{-\pi/2}B \subset \{z \leq 0\}$; we orient the angle such that $R_{\pi/2}B \subset \{z \leq 0\}$.
  - $R_\vartheta \partial B \cap W_S = \emptyset$ for all $\vartheta \in (0, \pi/2]$.
  - There exists $z_0 > 0$ such that $B \subset \{z < z_0\} =: X$, and we require that $R_\vartheta X \cap W_S$ be compact for all $\vartheta \in (0, \pi/2)$. Furthermore, we require that $R_{\vartheta_1} B \subset R_{\vartheta_2} X$ for all $\pi/2 \geq \vartheta_1 \geq \vartheta_2 \geq 0$.

When W_S is known from the context, we may sometimes also speak of B being in good position with respect to (rotation about) l.

It is immediate how to define B being in good position with respect to a convex wedge of a general half-space.

Definition II.2.3. Let E and B be two surfaces. We say that E (resp. ∂E) *lies above* B, if every point $p = (p_1, p_2, p_3) \in E$ (resp. ∂E) satisfies

$$p_3 > \max\{z \mid (p_1, p_2, z) \in B\} \cup \{-\infty\}.$$

Proposition II.2.4. *Let E be a properly immersed minimal surface contained in a convex wedge W of the half-space $\{z \geq 0\}$, and let B be a properly immersed minimal surface in good position with respect to W.*

If $R_\vartheta B \cap \partial E = \emptyset$ for all $\vartheta \in [0, \pi/2]$ (using notation as in definition II.2.2) and ∂E lies above B, then E lies above B.

Proof. First we remark that under our assumptions, ∂E is non-empty (by the half-space theorem).

We will use notation as in the definition above.
Define $B_\vartheta := R_\vartheta B$. Observe that $B_{\pi/2} \cap E = \emptyset$.

We claim that $B_\vartheta \cap E = \emptyset$ follows from the maximum principle for all $\vartheta \in [0, \pi/2]$.

Assume there exists a point $p \in E \cap B_{\vartheta_0}$ for some $\vartheta_0 \in (0, \pi/2)$.
Then $\vartheta_1 := \sup\{\vartheta \leq \pi/2 \mid B_\vartheta \cap E \neq \emptyset\}$ is positive.

Since $R_{\vartheta_1/2}X \cap W$ is compact, $E' := E \cap R_{\vartheta_1/2}X$ is a compact immersed minimal surface with boundary disjoint from B_{ϑ_1} (from our assumptions). However, we can conclude $E' \cap B_{\vartheta_1} \neq \emptyset$ by compactness of E'. Every point of intersection is a one-sided interior contact point, so the maximum principle shows $E \subset B_{\vartheta_1}$. This is a contradiction.

Now we know that $E \cap B_\vartheta = \emptyset$ for all $\vartheta \in (0, \pi/2]$; so $E \cap B = \emptyset$ follows from the maximum principle, which implies the claim (since B divides W into components by assumption). □

We turn our attention from minimal surfaces contained in a wedge of a half-space to minimal ends bounded by rays which are contained in a convex slab wedge, and prove the main result of this section:

Proposition II.2.5. *Let E be an end of a properly immersed minimal surface, which is bounded by rays l_1, l_2 and contained in a convex slab wedge W.*

Then E is asymptotic to H_{l_1,l_2} (in distance).

Proof. Without loss of generality, we make the following assumptions:

- E is contained in a horizontal slab.

- W is the smallest convex slab wedge containing ∂E (we can assume that by Theorem I.3.18).
- If l_1 and l_2 are parallel, we assume both to be parallel to the y-axis and contained in the xy-plane.
- l_2 is the image of l_1 under rotation of angle π in the y-axis.
- The ray l_1 starts at $(c, 0, h)$ (for some $c \geq 0, h \geq 0$), and has increasing x and y-values (for the last two assumptions, it may be necessary to cut off in a vertical plane).
- The boundary of E is contained in the xz-plane, except for l_1 and l_2 (by cutting in a vertical plane).

Observe that by definition, $\partial E \cap \{y = 0\}$ is compact; the rays l_1, l_2 are horizontal, because E is assumed to be contained in a horizontal slab.

From Theorem I.3.18, we deduce $E \subset \{y \geq 0\}$.

Since $\partial E \cap \{y = 0\} = E \cap \{y = 0\}$ is compact, there is some $b_0 > 0$ such that $E \cap \{y = 0\}$ is contained in the ball of radius b_0 about the origin.

Now we choose $b_1 \geq b_0$, such that $B_{b_1} + (0, 0, h)$ is in good position with respect to rotation about $\{y = 0, z = h + b_0\}$, and we conclude that E lies below $B_{b_1} + (0, 0, h)$ from Proposition II.2.4.

Using $-B_{b_1}$ to denote the reflection of B_{b_1} in the xy-plane, the same argument shows that E lies above $-B_{b_1} - (0, 0, h)$.

Hence, there is a number R_ε such that

$$E \cap \{y \geq R_\varepsilon\} \subset S_\varepsilon = \{-h - \varepsilon \leq z \leq h + \varepsilon\}$$

for any $\varepsilon > 0$ we choose.

Next, we show that $E \cap \{y \geq R\}$ (for some $R \geq R_\varepsilon$) is contained in the vertical convex slab wedge W_ε:

Take a standard catenoid C_1^W with vertical axis and waist circle contained in the plane $\{z = h\}$.

There is a number $R_{b_0} > 0$ such that if the axis of C_1^W is at least R_{b_0} from the xz-plane, the catenoid does not intersect the strip $\{(x, y, z) \,|\, y = 0, |z| < b_0\}$. In such a position, we can translate C_1^W parallel to the x-axis so far that it does not intersect W. By the maximum principle, C_1^W can be slided in (parallel to the x-axis) until it touches one of the rays. Choosing $R_r \geq R_{b_0}$ if necessary, we can make sure that sliding in C_1^W with axis having (large) positive x-values, the catenoid touches l_1 first.

To slide in C_1^W from the other direction (i.e. with axis having (large) negative x-values), we place C_1^W such that the waist circle is contained in $\{z = -h\}$. Then C_1^W touches l_2 first.

Choosing $R := \max(R_r, R_\varepsilon)$, we know that $E \cap \{y \geq R\}$ is contained in W_ε.

Thus, we have finished the proof if l_1 and l_2 are contained in a plane.

Otherwise, we choose ε such that $2h + 2\varepsilon$ is less than half the period D of $\tilde{H}_{l_1,l_2}$. Set $d := D/2$.

We consider $E \cap \{y \geq R\}$ only, and we assume $R = 0$ (by a translation).

Now, we consider the part G of $\bar{H}_{l_1,l_2}$ in the slab $\{-d/2 \leq z \leq d/2\}$. Observe that we have $G \supset H_{l_1,l_2}$ and $\partial G \subset \{y = C\}$ (for some $C < 0$); further, ∂G consists of two horizontal rays and a vertical segment.

We move G (by a translation parallel to the xy-plane) into a position in which the boundary of G is contained in the xz-plane, and its axis is contained in $\{x = D, y = 0\}$, where $D > 0$ is large enough to make this copy G' of G be in good position with respect to $W + (0, 0, \mathbb{R}^-)$ (and rotation about the line $\{y = 0, z = d/2\}$).

Since the boundary of $E \cap \{y \geq 0\}$ lies below G' by construction, we know that E lies below G' (from Proposition II.2.4).

Similarly, we find that E lies above $G'' := G' + (-2D, 0, 0)$.

The part in between G' and G'' (which E is contained in) becomes infinitely thin for lines $(x, y, \mathbb{R})$ with $(x, y, 0) \in W$ and $x^2 + y^2 \to \infty$.
This finishes the proof. □

Remark II.2.6. We believe that under the additional assumption of E having finite total curvature, the result above can be sharpened to say that E is C^1-asymptotic to H_{l_1,l_2}. To obtain this, one would have to show that the norm of the shape operator is bounded pointwise (The argument would be similar to some work by Schoen et al). Using this, one shows that E has a well-defined limit normal at the end, which implies the claim. ♢

II.3 Existence of certain minimal ends bounded by rays

In this section, we show that certain minimal properly immersed disks exist. Some of the surfaces discussed here will be the conjugate of half of a trinoid in the next chapter.

Definition II.3.1. Let Γ be a simple ray-ended curve consisting of two non-parallel rays l_1, l_2 and an arc joining the endpoints of the rays.

We define the *standard sequence* for Γ to be the sequence $(\Gamma_n)_{K<n\in\mathbb{N}}$ of Jordan curves obtained as follows:

Denote the axis of H_{l_1,l_2} by l. Let E_n be the plane parallel to l which intersects $\tilde{l}_1$ and $\tilde{l}_2$ in points having equal distance from l (in the direction of the ends of the rays) such that $d(l, E_n) = n$.

Denote the intersection point of E_n and $\tilde{l}_i$ by $p_{i,n}$ (for $i \in \{1, 2\}$).

Let K_0 be the least natural number such that $p_{1,n} \in l_1$ and $p_{2,n} \in l_2$.

For $n > K_0$ let Γ_n be the curve obtained from Γ by cutting off the ends along the arc joining $p_{1,n}$ and $p_{2,n}$ in $E_n \cap H_{l_1,l_2}$.

Let $K > K_0$ be the least number such that Γ_n is a Jordan curve for all $n > K$.

Lemma II.3.2. *If the sequence of solutions $(M_n)_{n\in\mathbb{N}}$ of the Plateau problem for a standard sequence $(\Gamma_n)_{n\in\mathbb{N}}$ (for some simple ray-ended curve Γ) exists and satisfies uniform local area bounds, then there is a minimal immersed disk M bounded by Γ of finite total curvature which is asymptotic to H_{l_1,l_2} (in distance).* [1]

Proof. Obviously, we can find a sequence of submanifolds N_n of $\mathbb{R}^3$ to which we can apply Theorem I.3.9; then Proposition II.2.5 applies (observe that the Γ_n have bounded total curvature; by Gauss-Bonnet, the same holds for the M_n and consequently for M). □

Theorem II.3.3. *Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of solutions of the Plateau problem for the standard sequence $(\Gamma_n)_{n\in\mathbb{N}}$ for some simple ray-ended curve Γ.*

If every M_n is embedded, the sequence $(M_n)_{n\in\mathbb{N}}$ satisfies uniform local area bounds.

Proof. Section II.4 is dedicated to the technical proof of this fact. □

Corollary II.3.4. *Let Γ be a simple ray-ended curve as before. If the solutions M_n of the Plateau problem for the standard sequence $(\Gamma_n)_{n\in\mathbb{N}}$ exist and are embedded, then there is a minimal immersed disk bounded by Γ of finite total curvature which is asymptotic to H_{l_1,l_2} (in distance).*

Proof. According to Lemma II.3.2, it suffices to show that the sequence $(M_n)_{n\in\mathbb{N}}$ satisfies uniform local area bounds; this is achieved by Theorem II.3.3. □

Proposition II.3.5. *Let l_1 and l_2 be two non-intersecting rays.*
Then there exists a minimal immersed disk Φ with boundary the two rays and the line segment joining the endpoints of the rays, which is contained in a convex slab wedge. Moreover, Φ is embedded, has finite total curvature and its end is C^1-asymptotic to H_{l_1,l_2}.

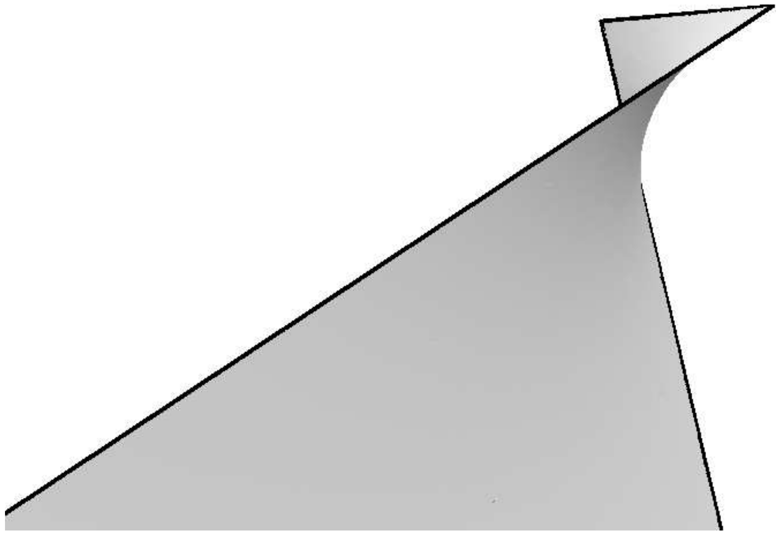

Figure II.2: A surface as constructed in Proposition II.3.5.

Proof. Note that the claim is trivial if the rays l_1, l_2 are contained in a plane; hence, we assume they are not; in particular, they are not parallel.

Define Γ to be the simple curve obtained from l_1, the segment connecting the endpoints of l_1 and l_2, and l_2.

Assume the ends of the rays l_1, l_2 to be symmetric to one another by rotation of angle π about the y-axis.
Let $h \in \mathbb{R}$ be such that $l_i \in \{z = (-1)^{i+1} h\}$.

In view of corollary II.3.4, we need only show that the Plateau solutions M_n for the standard sequence Γ_n are embedded (note that existence of the Plateau solutions is immediate).

We obtain this from Theorem I.3.4, where K_n is the compact convex subdomain of $\mathbb{R}^3$ bounded by the following planes (for n large enough):

$H_1 := \{z = -h\}$, $H_2 := \{z = h\}$, H_3 is the plane containing l_1 and the endpoint of l_2, and H_4 is the plane containing l_2 and the endpoint of l_1. Finally, $H_5 = E_n$.

This shows existence of a minimal disk Φ in a convex slab wedge, which has finite total curvature and is asymptotic to H_{l_1,l_2} in distance.

Let P be the plane containing the axis of H_{l_1,l_2}, and dividing H_{l_1,l_2} into two congruent pieces.[2]

Now every M_n is a graph over P by Lemma I.3.13; hence, so is Φ, which implies that it is embedded.

---

[1] Observe that $X_{\Gamma_{n_0}}$ being non-empty for some n_0 implies that there is a Plateau solution M_n for all n.

[2] Actually, P is the yz-plane under our assumptions.

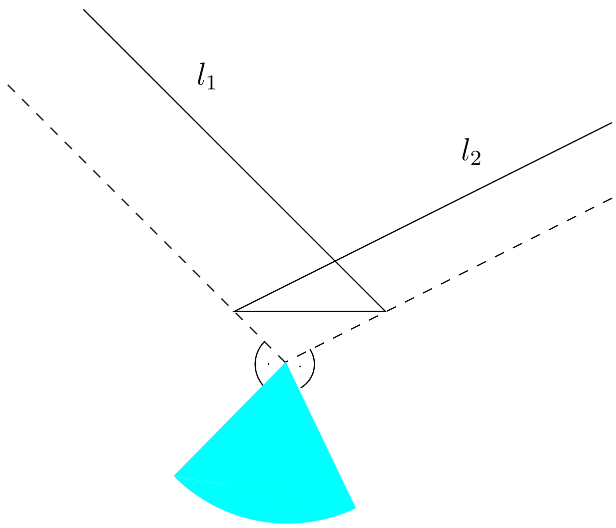

Figure II.3: The possible choices for L (projected to the xy-plane).

Since Φ is stable by construction, we obtain a (pointwise) curvature bound on Φ; using this, we conclude that Φ is C^1-asymptotic to H_{l_1,l_2}. □

Proposition II.3.6 (Uniqueness of the surfaces from Proposition II.3.5). *A surface Φ as constructed in Proposition II.3.5 is unique:*
If M is another minimal properly immersed disk which is contained in a convex slab wedge and $\partial M = \partial\Phi$, then $M = \Phi$.

Proof. We use notation as in the previous proof and assume the position of $\partial\Phi$ to be as we did there. Let r denote the closure of the segment $\Gamma\backslash(l_1 \cup l_2)$, and let C be the closure of the convex component of $\mathbb{R}^3\backslash(H_1 \cup H_2 \cup H_3 \cup H_4)$ containing $\partial\Phi$ in its boundary.

We note that M is contained in C (by Proposition I.3.17), and is asymptotic to H_{l_1,l_2} (in distance) by Proposition II.2.5.

The intersection $C \cap \{z = z_0\}$ is a sector bounded by rays parallel to l_1, l_2 resp. (for $z_0 \in [-h, h]$).

Now we choose a vertical line L such that rotating C about L has the following property: Let R_ϑ be the rotation of angle ϑ about L. We require that $R_\vartheta C \cap r = \emptyset = R_\vartheta r \cap C$ and $R_\vartheta l_i \cap l_i = \emptyset$ (for $i \in \{1, 2\}$) for all $\vartheta \in [-\pi, \pi]\backslash\{0\}$. This is possible by the previous remarks (cf. Figure II.3).

Now we consider the surface $\Phi_\vartheta := R_\vartheta\Phi$. We have $\Phi_{\pm\pi} \cap M = \emptyset$, and our knowledge about the asymptotics implies that $\Phi_\vartheta \cap M$ is compact for all $\vartheta \in [-\pi, \pi]\backslash\{0\}$. Thus, the maximum principle applies and $M = \Phi$. □

II.4 Uniform local area bounds for embedded standard sequences

In this section, we want to show that a sequence of embedded solutions of the Plateau problem for a standard sequence satisfies uniform local area bounds.

For three distinct points a, b, c in the plane, we denote by $\angle(a, b, c)$ the unoriented angle in $[0, \pi]$ enclosed by the rays through a (resp. c) starting at b.

We fix some $c_0 > 0$ and let $p_1 := (c_0, 0)$ and $p_2 := (-c_0, 0)$ be two points in $\mathbb{R}^2$.

For a point $p \in \mathbb{R}^2$, we denote by $K(p)$ the circle through p_1, p_2 and p (if p lies on the x-axis, $K(p)$ is the x-axis).

We start with some technical Lemmas:

Lemma II.4.1. *The function* $f : \mathbb{R}^2 \to \mathbb{R}, \quad p \mapsto \|p\| \cdot \angle(p_1, p, p_2)$ *is bounded (on* $\mathbb{R}^2 \backslash \{p_1, p_2\}$*).*

Proof. Note that the angle does not change when we move p along the circle $K(p)$ (as long as we do not pass through p_1 or p_2). Along this arc of a circle, $\|p\|$ attains its maximum on the y-axis (for small angles, i.e. large values of $\|p\|$, which is the interesting case; note that the points on the x-axis need not be considered, since the angle is 0 for points $(x, 0)$ with $|x| > c_0$).

Hence, we consider the function $g : \mathbb{R}^{\geq 0} \to \mathbb{R}$, $g(r) = r \cdot \angle(p_1, (0, r), p_2)$ and show that it is bounded, which is the case if we can prove that it converges to some finite value as r goes to infinity.

We have

$$\angle(p_1, (0, r), p_2) = 2 \tan^{-1}\left(\frac{c_0}{r}\right)$$

So we calculate, using L'Hospital's rule:

$$\lim_{r \to \infty} \frac{\tan^{-1}(c_0 r^{-1})}{r^{-1}} = \lim_{r \to \infty} \frac{\frac{1}{1 + c_0^2 r^{-2}}(-c_0 r^{-2})}{-r^{-2}} = c_0 \qquad \square$$

Observe that the function is bounded on $\mathbb{R}^2$ when its domain is extended. However, such an extension would not be continuous.

Definition II.4.2. We call two helicoids *parallel* if their axes are parallel and one can be obtained from the other by a translation, where the translation vector is orthogonal to their axes.

Now consider two non-parallel rays l_1, l_2 in $\mathbb{R}^2$ emanating from the same point (d, e). We assume that l_1, l_2 point in the direction with large y-values, and that reflection in the line $\{x = d\}$ maps l_1 onto l_2.

Setting $l := (d, e, \mathbb{R})$, we consider planes E_t as in the definition of a standard sequence (for $t > 0$; in our case $E_t = \{y = e + t\}$).

Let S' be the sector of $\mathbb{R}^2$ determined by l_1, l_2 of convex angle, and let $S := S' \times \mathbb{R}$.

Using this notation, we have:

Lemma II.4.3. *Consider two parallel helicoids H_i with axes $(p_i, \mathbb{R})$ (for $i \in \{1, 2\}$).*

Then $(S \cap E_t) \backslash (H_1 \cup H_2)$ consists of two (congruency) classes of connected components ("strips") of areas $A_1(t) \leq A_2(t)$, and A_1 is bounded (for $t > \max\{0, -e\}$).

Proof. The last part of the statement is the only one which requires proof.

On $\{(x, y) \in \mathbb{R}^2 \,|\, y > e\}$ let $l(x, y)$ be the length of the segment $E_t \cap (S' \times \{0\})$, with t such that $(x, y, 0) \in E_t$. Then l is an affine function of y.

Note that the smaller height difference between the two helicoids along a line $(x, y, \mathbb{R})$ is a linear function of the angle $\angle(p_1, (x, y), p_2)$ (where the proportion depends on the parameter of the helicoids (cf. example I.2.5)).

By these remarks, it suffices to show that the function $y \cdot \angle(p_1, (x, y), p_2)$ is bounded on $\{y > e\}$.

This follows from the previous Lemma since $\|(x, y)\| \geq y$. □

Next, we need some information about the relationship of area on a helicoid compared with the area of the plane orthogonal to its axis.

Lemma II.4.4. *Let $G \subset H_\lambda$ be a connected part of the helicoid H_λ, which is graph over the disk of radius $R > 0$ in the xy-plane.*
Then the area of G is

$$\operatorname{area}(G) = \pi\lambda^2(e^{2r} - e^{-2r} + 4r)$$

where $2|\lambda| \sinh r = R$.

Proof. It follows from Lemma I.1.5 and example I.2.5 that we can compute the area of G by integrating $\lambda^2(e^{2x} + e^{-2x} + 2) = 4\lambda^2 \cosh^2 x$ over the rectangle $[0, r] + i[0, 2\pi]$, where $2|\lambda| \sinh r = R$. We compute

$$\begin{aligned}\operatorname{area}(G) &= 2\pi\lambda^2 \int_0^r e^{2x} + e^{-2x} + 2\,dx = \pi\lambda^2[e^{2x} - e^{-2x} + 4x]_0^r \\ &= \pi\lambda^2(e^{2r} - e^{-2r} + 4r)\end{aligned}$$

□

Lemma II.4.5. *Let $G' \subset \mathbb{C}$ be the rectangle $[r_0, r_0 + \Delta r] + i[y_0, y_0 + 2\pi]$ for some $y_0 \in \mathbb{R}$ and $r_0, \Delta r \geq 0$. Let G_{pl} be the projection of $G := H_\lambda(G')$ to the xy-plane. Then we have*

$$\text{area}(G) \leq \text{area}(G_{pl}) + \bar{K}_{\Delta r,\lambda}$$

for a constant $\bar{K}_{\Delta r,\lambda}$ depending on Δr and λ only.

Proof. First we observe that G_{pl} is an annulus with $2|\lambda| \sinh(r_0 + \Delta r)$ as the outer radius and inner radius $2|\lambda| \sinh r_0$. Hence

$$\begin{aligned}\text{area}(G_{pl}) &= \pi 4\lambda^2(\sinh^2(r_0 + \Delta r) - \sinh^2 r_0) \\ &= \pi\lambda^2(e^{2(r_0+\Delta r)} - e^{2r_0} + e^{-2(r_0+\Delta r)} - e^{-2r_0}) \\ &\geq \pi\lambda^2(e^{2(r_0+\Delta r)} - e^{2r_0} - 1)\end{aligned}$$

Using the previous Lemma, we compute the area of G:

$$\begin{aligned}\text{area}(G) &= \pi\lambda^2(e^{2(r_0+\Delta r)} - e^{2r_0} - e^{-2(r_0+\Delta r)} + e^{-2r_0} + 4\Delta r) \\ &\leq \pi\lambda^2(e^{2(r_0+\Delta r)} - e^{2r_0} + 1 + 4\Delta r)\end{aligned}$$

Hence, the claim holds when we set $\bar{K}_{\Delta r,\lambda} := \pi\lambda^2(4\Delta r + 2)$. □

Corollary II.4.6. *The claim of the previous Lemma holds if G' is a subset of a rectangle $K := [r_0, r_0 + \Delta r] + i[y_0, y_0 + 2\pi]$.*

Proof. This is immediate since the area of the helicoid parametrized by $K \backslash G'$ is greater than the area of its projection. □

Corollary II.4.7. *Fix $C > 1, \lambda \neq 0$. There are $R_{C,\lambda} > 0$ and $K_{C,\lambda}$ such that the following holds:*

Let $r > R_{C,\lambda}$, and let G be a connected subset of H_λ which is graph over the annulus G_{pl} in the xy-plane centered at the origin with inner radius r and outer radius $C \cdot r$. Then

$$\text{area}(G) \leq \text{area}(G_{pl}) + K_{C,\lambda}$$

Proof. We want to apply the previous corollary (II.4.6) with $\Delta r = 1$ and $K_{C,\lambda} = \bar{K}_{1,\lambda}$. The existence of a suitable constant $R_{C,\lambda}$ follows from the fact that the difference of outer and inner radius of the annuli considered up to now grow exponentially with r. □

Corollary II.4.8. *The claim of the previous corollary holds for connected subsets of H_λ which are graph over part of annuli $[r, C \cdot r] \cdot \exp(i[0, 2\pi])$ (considered as subsets of the xy-plane).*

Proof. Exactly as the proof of II.4.6. □

Now we have all the ingredients to prove Theorem II.3.3:

Theorem II.4.9 (= II.3.3). *Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of solutions of the Plateau problem for the standard sequence $(\Gamma_n)_{n\in\mathbb{N}}$ for some simple ray-ended curve Γ.*

If every M_n is embedded, the sequence $(M_n)_{n\in\mathbb{N}}$ satisfies uniform local area bounds.

Proof of Theorem II.3.3. Denote the rays of Γ by l_1, l_2 and assume the rays l_1, l_2 to be horizontal and symmetric to one another by rotation of angle π about the y-axis.

Further, we assume

$$0 = \sup\{z \mid \exists(x, y).(x, y, z) \in \Gamma\backslash(l_1 \cup l_2)\}.$$

Just as in the proof of II.2.5, we obtain two copies G' and G'' of a part G of $\tilde{H}_{l_1,l_2}$ such that all the $M_n \cap \{y \geq 0\}$ are contained in between G' and G''; we have $G' = G'' + (C, 0, 0)$ for some $C \in \mathbb{R}$ and $\partial G' \subset \{y = R\}$ for some $R \in \mathbb{R}$.

Denote the axis of H_{l_1,l_2} by l; then l is $(d, e, \mathbb{R})$ for some $d, e \in \mathbb{R}$.

Observe that the E_n from the definition of a standard sequence are $E_n = \{y = e + n\}$.

Let $n_0 = \lceil\max\{1, 1 + R - e\}\rceil$. The areas of M_n with $n \leq n_0$ can be bounded by their maximum. Fix an $n > n_0$; we show that M_n has bounded area locally, and the bound we find will not depend on n.

From Lemma II.4.3, we obtain a number C_1 such that $M_n \cap \{y = m\}$ is contained in a "strip" of area at most C_1 for all $m \geq n_0$ (observe that the strip between G' and G'' is the smaller strip in the sense of Lemma II.4.3).

Now the area of M_n in $\{y \leq e + n_0 + 1\}$ is bounded by $|M_{n_0+1}| + C_1$: The intersection $M_n \cap E_{n_0+1}$ is a simple curve by Lemma II.1.2 (since M_n is embedded). Hence, we obtain a comparison surface of the type of the disk with the same boundary, and area at most $|M_{n_0+1}| + C_1$, and the area of $M_n \cap \{y \leq e + n_0 + 1\}$ is less than this because M_n is area-minimizing.

Similar to the technique we used in the proof of II.1.5, we consider cylinders $C(p)$ of radius $1/2$ with vertical axis containing p.

We want to show that $|M_n \cap C(p)|$ is bounded for all p with y-coordinate at least $e + n_0 + 1/2$, which will finish the proof.

Every such cylinder is contained between two planes E_t and E_{t+1} (where t is not a natural number in general).

For the moment, we assume that the y-coordinate of p is at most $e+n-1/2$.

Now $M_n \cap C(p)$ is contained in $W_\varepsilon \cap \{t \leq y \leq t + 1\} =: K$.

Let A_h be the part of H_{l_1,l_2} contained in K, and let $A_{pl} = K \cap \{z = 0\}$ be the projection of A_h to the xy-plane.

Constructing a comparison surface, we obtain

$$\text{area}(M_n \cap K) \leq \text{area}(A_h) + 2C_1.$$

Denoting half of the opening angle of W_ε by α, we observe that A_{pl} is contained in an annulus centered at $(d, e, 0)$ of inner radius t and outer radius $(t+1)/\cos\alpha \leq (2/\cos\alpha) \cdot t$ (since $t \geq 1$ by our assumption on n_0).

Hence, we obtain

$$\text{area}(A_h) \leq \text{area}(A_{pl}) + C_2$$

for some constant C_2 independent from t from Corollary II.4.8 (increasing n_0 if necessary[3]).

Now we have

$$\text{area}(M_n \cap K) \leq \text{area}(A_{pl}) + 2C_1 + C_2.$$

Note that M_n intersects every vertical line passing through K at least once (otherwise, M_n is not of the type of a disk); hence we have:

$$\text{area}(M_n \cap (K \backslash C(p))) \geq \text{area}(A_{pl} \backslash C(p)) \geq \text{area}(A_{pl}) - \pi/4$$

Now we can conclude

$$\text{area}(M_n \cap C(p)) \leq 2C_1 + C_2 + \pi/4.$$

Lastly, we drop the assumption that the y-coordinate of p be less than $e + n - 1/2$: If the y-coordinate is greater than $e + n + 1/2$, there is nothing to show; otherwise, $M_n \cap C(p)$ is contained between E_{n-1} and E_n, and the argument works precisely as given above. □

[3]Observe that whether n_0 needs to be increased here or not is independent from n; we could have taken the condition into account when defining n_0, but this would have made the proof even more technical.

Chapter III

Trinoids and the Classifying Map

III.1 Trinoids

Definition III.1.1. An (annular) properly immersed end of a Bryant surface is called *catenoidal* if it is embedded and asymptotic to a catenoid cousin.

Definition III.1.2. We define a *trinoid* to be a properly immersed Bryant surface of genus zero with three ends, all of which are catenoidal. Further, we require that a trinoid be symmetric by reflection in any hyperbolic plane containing its asymptotic boundary.

Denote by $\mathcal{M}$ the space of trinoids with ends marked by 1, 2, 3, up to isometry (respecting the marks of the ends).

In this paper, we want to exhibit some properties of the space of trinoids. One should mention that there is a Theorem in [UY00, Thm. 2.6] which classifies a slightly different class of trinoids (but the conditions in their Theorem are fairly abstract). Also, in [BPS02], formulas for the trinoids classified by Umehara-Yamada are calculated.

However, the approach presented in this thesis using the relation of CMC-1-surfaces in $\mathbb{H}^3$ to minimal surfaces in $\mathbb{R}^3$ is different and we believe that one gets a good idea of the geometry of the surfaces from our methods.

The idea of studying trinoids in the way described here stems from a talk given by Harold Rosenberg at the "Workshop on Minimal Surfaces" in Bonn, June 2000.

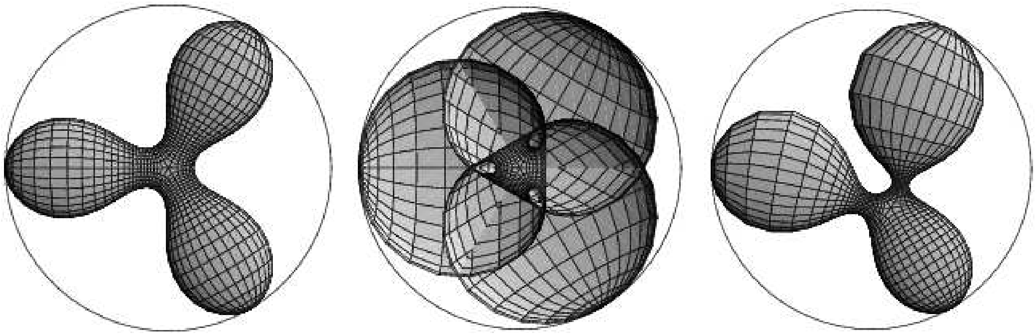

Figure III.1: An embedded symmetric trinoid, a non-embedded symmetric trinoid, and a non-symmetric embedded trinoid.[1]

First, we exhibit some general properties of a trinoid.

Note that a trinoid is a map $S^2 \setminus \{x_1, x_2, x_3\} \to \mathbb{H}^3$, where x_1, x_2, x_3 are distinct and correspond to the ends 1, 2, 3 respectively.

Our first observation is:

Lemma III.1.3. *Any properly embedded Bryant surface of genus zero with three ends is a trinoid.*

Proof. Let M be a properly embedded Bryant surface of genus 0 with three ends. By Theorem I.8.2, every end is catenoidal, and by [CHR01, Thm. 11], the three asymptotic boundary points are distinct and M is a bigraph over the plane containing them. □

We believe that the Lemma above generalizes to Alexandrov-embedded surfaces.

Lemma III.1.4. *The ends x_i, $i \in \{1, 2, 3\}$ of a trinoid M correspond to distinct points of $\partial\mathbb{H}^3$. Hence, the symmetry plane of M obtained from the definition is unique.*

Proof. First, we observe that every properly embedded end of a CMC-1-surface in $\mathbb{H}^3$ is asymptotic to precisely one point in $\partial\mathbb{H}^3$ by Theorem I.8.1 (actually, this is one of the main ingredients in the proof of this Theorem).

If all ends correspond to the same point in $\partial\mathbb{H}^3$, then the trinoid M is contained in a horosphere (by the large amount of symmetry it has). This is a contradiction, since no end is asymptotic to a horosphere by assumption.

[1]We present these pictures with the friendly permission of A. Bobenko, T. Pavlyukevich and B. Springborn. The pictures, which accompany [BPS02], are available online at http://www-sfb288.math.tu-berlin.de/~bobenko/Trinoid/webimages.html

Assume that the ends correspond to two distinct points at infinity only.

Without loss of generality, let x_1, x_2 correspond to the two asymptotic boundary points of M. From the assumption on the symmetry, M is a surface of revolution about the geodesic joining $M(x_1), M(x_2)$. Choose any path γ in $S^2\backslash\{x_1, x_2, x_3\}$ joining x_1, x_2. Since M is a surface of revolution, the rotated image of γ is part of M.

The domain of a connected properly immersed surface of revolution with embedded ends is a sphere without two points. This contradiction shows that the three ends of M correspond to three distinct points in $\partial\mathbb{H}^3$. These three points lie in the boundary of a unique plane of $\mathbb{H}^3$. □

Lemma III.1.5. *Let $\Phi : \Sigma\backslash\{p_j\}$ be a properly immersed Bryant surface.*

Then two principal geodesic segments intersecting in $x \in \Sigma\backslash\{p_j\}$ which are mapped into the same plane P coincide (in a neighborhood of x).

Proof. If the two segments intersect tangentially in x, they coincide by uniqueness of geodesics.

They cannot intersect transversally, since Φ intersects P orthogonally. □

Lemma III.1.6. *Let $\Phi : S^2\backslash\{p_j \,|\, 1 \leq j \leq n\} \to \mathbb{H}^3$ be a properly immersed Bryant surface. Let l be an embedded loop in S^2 comprising points from the set $\{p_j \,|\, 1 \leq j \leq n\}$ and principal geodesics mapped to a hyperbolic plane P. Then reflecting Φ in P exchanges the two components of $S^2\backslash l$.*

Proof. It follows from the assumption that $S^2\backslash l$ consists of two components.

Near any point of l on a principal geodesic, reflection in P exchanges the two components of $S^2\backslash l$. Since reflection in P induces a homeomorphism of $S^2\backslash\{p_j\}$, the claim follows. □

Lemma III.1.7. *For a trinoid $M \in \mathcal{M}$, there is a unique principal geodesic of M joining x_1 to x_2, which we denote by l_{12}. Similarly, there is a unique principal geodesic l_{23} joining (x_2, x_3) and a unique principal geodesic l_{31} joining (x_1, x_2).*
Considering l_{12}, l_{23}, l_{31} as subsets of S^2, we have that $S^2\backslash(l_{12} \cup l_{23} \cup l_{31} \cup \{x_1, x_2, x_3\})$ consists of exactly two components.

Proof. We know that a principal geodesic is contained in a plane of symmetry of M. First we try to find l_{12}. The symmetry plane of M which l_{12} needs to be contained in (if it exists) needs to fix $M(x_1)$ and $M(x_2)$. Since the three ends of M are distinct, the only symmetry plane P of that kind contains $M(x_3)$ as well (so P is necessarily the symmetry plane of M from the definition and Lemma III.1.4).

Every end of M is embedded by assumption, so there are two curves of intersection between P and M near x_1. These are principal geodesics (by remark I.7.6) and can be continued to maximal geodesics contained in P, which are principal geodesics by Lemma I.7.7.

Let l be such a maximal principal geodesic. The image of l in S^2 has no self-intersections by Lemma III.1.5. Observe that l starts and ends at points from the set $\{x_1, x_2, x_3\}$ (and at least one endpoint is x_1).

We claim that the two endpoints of l are not the same. Indeed, assume that the two endpoints were the same. Then l is a loop in S^2 starting and ending at x_1. By Lemma III.1.6 the components of $S^2 \backslash (l \cup \{x_1\})$ are exchanged by reflection of M in P. This is a contradiction, since x_2 and x_3 are fixed points of this reflection.

Hence l starts at x_1 and ends at x_2 (or x_3). The other choice for l above yields a principal geodesic l'. Now l and l' do not intersect in S^2 (by Lemma III.1.5), and cannot end at the same point (otherwise, we obtain a contradiction to Lemma III.1.6 as above). Thus, $\{l, l'\} = \{l_{12}, l_{31}\}$ from the claim.

We have shown existence and uniqueness of l_{12}, l_{23}, l_{31}. By Lemma III.1.5, their (pairwise) intersection in S^2 is empty. □

Lemma III.1.8. *Consider a trinoid M and its symmetry plane P from Lemma III.1.4. Then there is a neighborhood N of $l_{12} \cup l_{23} \cup l_{31} \cup \{x_1, x_2, x_3\}$ in S^2 such that $M(N) \cap P = M(l_{12} \cup l_{23} \cup l_{31})$. In particular, each component of $S^2 \backslash (l_{12} \cup l_{23} \cup l_{31} \cup \{x_1, x_2, x_3\})$ is mapped to a component of $\mathbb{H}^3 \backslash P$ near its boundary.*

If M is embedded, each component of $S^2 \backslash (l_{12} \cup l_{23} \cup l_{31} \cup \{x_1, x_2, x_3\})$ is mapped into a halfspace of $\mathbb{H}^3 \backslash P$.

Proof. The first part of the claim is immediate from Lemma III.1.6 and the fact that the ends of M are embedded.

If one component M' of $S^2 \backslash (l_{12} \cup l_{23} \cup l_{31} \cup \{x_1, x_2, x_3\})$ is not mapped into a halfspace of $\mathbb{H}^3 \backslash P$, then M has a self-intersection in the set $M' \cap P$ (which is non-empty). □

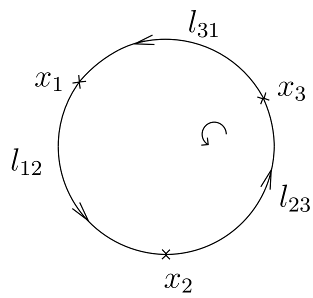

Figure III.2: The domain of M^+ and M^c.

III.2 The classifying map

By definition, every end of a trinoid is asymptotic to C_λ for some $\lambda \in I$, where $I := (-\frac{1}{4}, \infty)\backslash\{0\}$.

Let $J := (0, \infty)\backslash\{\pi\}$, and recall the bijective function $\tilde{\varphi} : I \to J$ defined by $\tilde{\varphi}(\lambda) := \frac{\pi}{\sqrt{1+4\lambda}}$ in example I.5.8.

Definition III.2.1. We can define the map $\Psi : \mathcal{M} \to J^3$ sending a trinoid to the triple $(\varphi_1, \varphi_2, \varphi_3) \in J^3$, where $\varphi_i = \tilde{\varphi}(\lambda_i)$ and λ_i is the parameter of end i.

Given a trinoid M, we can (by an orientation-preserving isometry) assume that its symmetry plane is the equatorial plane E of the Poincaré disk model. Further, we can assume that the ends are marked increasingly if one looks from above (where we fix a direction as "above" once and for all).

Definition III.2.2. Given a trinoid M, we divide its domain $S^2\backslash\{x_1, x_2, x_3\}$ into two components along l_{12}, l_{23}, l_{31} (cf. Lemma III.1.7), and we define M^+ to be the restriction of M to the closure of the component which is mapped to the upper half space near l_{12}, l_{23}, l_{31}, if M is put in the Poincaré model in the way explained above (cf. Lemma III.1.8).
Note that M^+ is well-defined up to orientation-preserving isometries leaving the upper half-space in the Poincaré disk model invariant.
Denote by $M^c := (M^+)^c$ the conjugate surface of M^+ (cf. section I.9).

So M^+ is a map $\bar{D}\backslash\{x_1, x_2, x_3\} \to \mathbb{H}^3$, where x_1, x_2, x_3 are distinct points in ∂D.
We choose the orientation on D and its boundary as depicted in Figure III.2.

Lemma III.2.3. *Let M be a trinoid. Then $M^c = (M^+)^c$ is a minimal surface bounded by three straight lines (which are monotonically parametrized).*

Proof. Since l_{12}, l_{23}, l_{31} are principal geodesics (of infinite length), this follows from Proposition I.9.2 (and the assumption of M being immersed). $\square$

Denote the boundary lines of M^c by $\tilde{l}_{12} := M^c(l_{12})$, $\tilde{l}_{23} := M^c(l_{23})$ and $\tilde{l}_{31} := M^c(l_{31})$.

Definition III.2.4. For a real number φ, we define $r(\varphi) := \min_{n\in\mathbb{Z}} |\varphi + 2n\pi|$ as the *reduced angle* of φ.

Lemma III.2.5. *The lines $\tilde{l}_{31}$ and $\tilde{l}_{12}$ (of ∂M^c) are contained in $I_1 H_{\lambda_1}$ for some orientation-preserving isometry I_1 of $\mathbb{R}^3$. In particular, the angle between the ends of $\tilde{l}_{31}$ and $\tilde{l}_{12}$ which are adjacent to end 1 is $r(\varphi_1)$ ($= r(\tilde{\varphi}(\lambda_1))$). The distance of these two lines is $h(\varphi_1) := |\lambda_1|\varphi_1$.*

Corresponding statements hold for the other ends.

Proof. First we note that $h : J \to \mathbb{R}$ defined as above is well-defined, because $\tilde{\varphi}$ is a bijective function (in fact $h(\varphi) = |\frac{\pi^2}{4\varphi} - \frac{\varphi}{4}|$).

Since we know that any end of M^+ is asymptotic to half of a catenoid cousin C_λ, the corresponding end of M^c is asymptotic to (some copy of) the helicoid H_λ (by I.9.3). By the remark at the end of example I.5.8, we have that $\Delta y = \tilde{\varphi}(\lambda)$ is the change of angle necessary to obtain half of a catenoid cousin.

On the helicoid, Δy is the angle between the corresponding lines (when rotated inside the helicoid).

The distance of the two lines can be seen to be $|\lambda|\Delta y$ from the formulas given in example I.2.5. $\square$

Lemma III.2.6. *Every trinoid $M \in \mathcal{M}$ has finite total curvature.*

Proof. This is immediate since every catenoid cousin has finite total curvature (and the asymptotics is exponential). $\square$

Lemma III.2.7 (Period Problem). *The conditions from Lemma III.2.5 are sufficient to solve the period problem.*
More explicitly:
Consider a properly immersed minimal disk $X : \bar{D}\backslash\{x_1, x_2, x_3\} \to \mathbb{R}^3$ which sends every boundary component to a (monotonically parametrized) straight line. If every end E of X is C^1-asymptotic to a segment of the helicoid H_{λ_E} and the conditions from Lemma III.2.5 are satisfied, then X gives rise to a trinoid $\bar{X} : S^2\backslash\{x_1, x_2, x_3\} \to \mathbb{H}^3$ with $\Psi(\bar{X}) = (\tilde{\varphi}(\lambda_1), \tilde{\varphi}(\lambda_2), \tilde{\varphi}(\lambda_3))$.

Proof. It suffices to show that the conjugate surface $\tilde{X}$ of X sends all boundary components into one hyperbolic plane H. If that is the case, we obtain $\bar{X}$

from $\tilde{X}$ by a reflection in H. This will be a CMC-1-surface (by the Schwarz reflection principle applied to X).

Denote the boundary components of $\bar{D}\backslash\{x_1, x_2, x_3\}$ by l_{12}, l_{23}, l_{31} as before. We know from I.9.2 that l_{12} is contained in a plane H_{12}, and l_{23} is contained in a plane H_{23}, and lastly l_{31} is contained in a plane H_{31}, and that $\tilde{X}$ can be extended by reflection in these planes.

Rotating X about l_{12} maps H_{λ_1} to itself. Hence, the symmetry plane of X's associate surface corresponding to l_{12} is a symmetry plane for $C^W_{\lambda_1}$, the corresponding asymptotic surface. This symmetry plane is mapped by the Bryant cousin relation to a symmetry plane of the Catenoid cousin C_{λ_1}, which is H_{12} (by I.7.5).

Since any end of $\tilde{X}$ is asymptotic to half of a catenoid cousin by construction, any two of the three planes must agree. □

We want to mention another interesting fact about M^+:

Lemma III.2.8. *If M^+ is embedded, then so is M.*

Proof. Assume the symmetry plane of M to be the equatorial plane E as before.

We apply the Alexandrov-reflection technique:[2] Using a (continuous) family of planes which foliate the upper half-space, we conclude that the normal of M^+ at any point $p \in M^+ \cap E$ has non-positive vertical coordinate.

Similarly, we use a family of planes foliating the lower half-space to find that for a point $p \in M^+ \cap E$, the normal has to have non-negative vertical coordinate.

Thus, every component of $M \cap E$ is a principal geodesic, i.e. a curve of planar reflection (by I.7.3). So M^+ is cut off wherever it reaches E, and it does not intersect the lower half-space; so M is embedded. □

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[2] A well-readable introduction to this technique can be found in [LR85].

III.3 Necessary conditions on the constellation of boundary lines

In this section, we use the information about the constellation of lines which bound M^c to obtain a necessary condition on the parameter triple $\Psi(M)$.

Definition III.3.1. A triple $(\lambda_1, \lambda_2, \lambda_3) \in I^3$ is called a *parameter triple*.

A triple of oriented lines (l_{12}, l_{23}, l_{31}) in $\mathbb{R}^3$ is called *admissible constellation* if

(i) There exists a parameter triple $(\lambda_1, \lambda_2, \lambda_3) \in I^3$ and orientation-preserving isometries I_1, I_2, I_3 of $\mathbb{R}^3$ such that

$$\{l_{12}, l_{31}\} \subset I_1(H_{\lambda_1}), \quad \{l_{12}, l_{23}\} \subset I_2(H_{\lambda_2}), \quad \text{and } \{l_{23}, l_{31}\} \subset I_3 H_{\lambda_3}.$$

(ii) Rotating l_{12} inside $I_1(H_{\lambda_1})$ maps l_{12} to l_{31} with the opposite orientation (similarly for the other pairs of lines).

(iii) The distance of l_{31} and l_{12} is $h \circ \tilde{\varphi}(\lambda_1)$, and similarly for the other pairs of lines.

A triple $(\varphi_1, \varphi_2, \varphi_3) \in J^3$ of angles is called *admissible*, if there exists an admissible constellation with parameter triple $(\tilde{\varphi}^{-1}(\varphi_1), \tilde{\varphi}^{-1}(\varphi_2), \tilde{\varphi}^{-1}(\varphi_3))$.

An admissible triple is called *non-parallel* if there is a corresponding admissible constellation such that the lines are not contained in parallel planes. The triple is called *parallel* otherwise.

We define $\mathcal{T}$ to be the set of interior points of the tetrahedron with vertices $(\pi, 0, 0)$, $(0, \pi, 0)$, $(0, 0, \pi)$, (π, π, π).

Observe that condition (iii) above is equivalent (by (i)) to the following:

(iii') The angle of rotation rotating l_{31} to l_{12} in $I_1(H_{\lambda_1})$ is $\tilde{\varphi}(\lambda_1)$, and similarly for the other pairs of lines.

Lemma III.3.2. *For any trinoid $M \in \mathcal{M}$, the triple $\Psi(M)$ is admissible.*

Proof. Clearly, we obtain an admissible constellation for $\Psi(M)$ from the boundary of M^c (cf. the proof of Lemma III.2.5). Note that in particular, the definition of an admissible triple corresponds to the orientation we chose on the domain of M^c (see Figure III.2). □

Proposition III.3.3. *The set of admissible non-parallel triples in $(0, \pi)^3$ is $\mathcal{T}$. For every admissible triple in $(0, \pi)^3$, there are exactly two non-parallel admissible constellations of lines in $\mathbb{R}^3$ (modulo $SO(3)$).*

Proof. Note that for a triple being inside the tetrahedron means satisfying the following four inequalities:

$$\sum \varphi_i > \pi \tag{III.3.1}$$

$$\varphi_1 + \varphi_2 - \varphi_3 < \pi \tag{III.3.2}$$

$$\varphi_1 - \varphi_2 + \varphi_3 < \pi \tag{III.3.3}$$

$$-\varphi_1 + \varphi_2 + \varphi_3 < \pi \tag{III.3.4}$$

Without loss of generality, we consider triples which satisfy $(\pi >)\ \varphi_1 \geq \varphi_2 \geq \varphi_3$ only. Note that in this case, we need to prove that III.3.1 and III.3.2 are necessary and sufficient for admissible constellations to exist.

For given angles $\varphi_1 \geq \varphi_2 \geq \hat{\varphi}_3$, we will examine whether an admissible constellation exists.

Let the line l_{12} be $t \mapsto (t, 0, 0)$ and l_{31} be $t \mapsto (-t\cos\varphi_1, h(\varphi_1), -t\sin\varphi_1)$ (contained in a plane parallel to the xz-plane). By construction, l_{12} and l_{31} satisfy the conditions regarding their relative constellation. Observe that any admissible constellation for any triple $(\varphi_1, \bullet, \bullet)$ can be moved by an orientation-preserving isometry such that l_{12} and l_{31} are in the positions as described here (compare the formula for H_λ computed in example I.2.5).

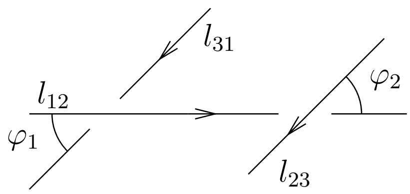
In order to find l_{23}, we use the cylinder C_2 around l_{12} of radius $h_2 := h(\varphi_2)$ and the cylinder C_3 around l_{31} of radius $h(\hat{\varphi}_3)$. Then l_{23} has to be tangent to these two cylinders.

For later use, we give a parametrization of C_2:

$$C_2(x, \vartheta) = (x, h_2\cos\vartheta, h_2\sin\vartheta)$$

We examine what a line l_{23} which has an angle of φ_2 with l_{12} and is tangent to C_2 has to look like.

For any given point on C_2, there are two ways how l_{23} could be tangent to C_2 in that point (the orientation is fixed by (ii) of the definition of an admissible constellation). The following way can be eliminated because l_{12} and l_{23} have to lie in some copy of H_{λ_2} (and in the following picture, they could only be contained in $H_{-\lambda_2}$):[3]



[3]One should remark at this point that the sketches in this proof are projections to the plane xz-plane.

From now on, we consider constellations of lines with l_{12} and l_{31} as given, and l_{23} some line tangent to C_2 and C_3, and such that l_{23} and l_{12} are contained in a copy of H_{λ_2}.

Up to varying l_{23} continuously under these constraints, there are four essential possibilities of the relative position of the lines left:

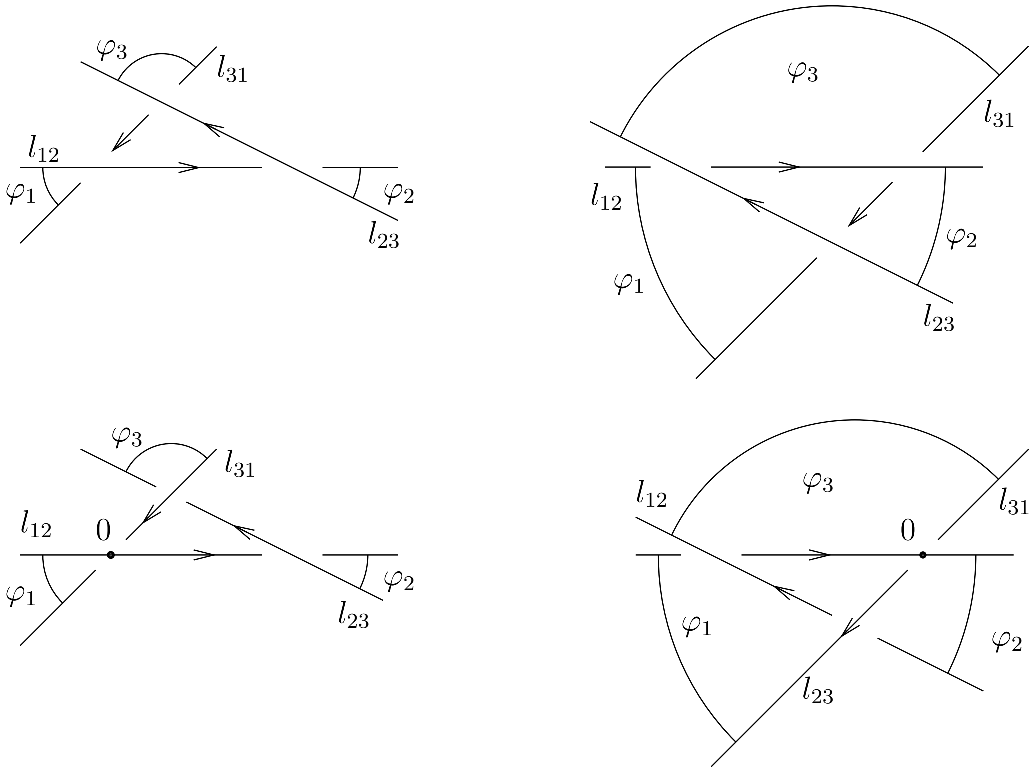


Figure III.3: Some constellations.

The upper two cases are ruled out since l_{31} and l_{23} have to be contained in some copy of H_{λ_3}.

The two constellations which are left differ by the orientation of the lines only (and by a rotation of π about the y-axis).

So up to varying l_{23} continuously under the given constraints, it suffices to consider one of these constellations (and we pick the left one).

We examine whether $\hat{\varphi}_3$ can occur as angle φ_3 between l_{23} and l_{31}.

First, we note that for every value of ϑ (with the possible exception of $\vartheta \in \mathbb{Z}\pi$) in the parametrization of C_2 given above, there is exactly one point (x, ϑ) on the cylinder such that l_{23} tangent to C_2 in $C_2(x, \vartheta)$ is also tangent to C_3 in one of the possible constellations left.

Our previous remark about equivalence of the left-hand and right-hand side of the figure above implies that we can restrict our attention to the case $\vartheta \in (0, \pi)$ (we can ignore $0, \pi$ - if they can occur - by the assumption of non-parallelity).

To compute the angle φ_3, we use the (negative) tangent vector of l_{23} tangent to a point (x, ϑ), which is $v_\vartheta := (\cos\varphi_2, -\sin\varphi_2 \sin\vartheta, \sin\varphi_2 \cos\vartheta)$.

So the cosine of φ_3 is given by:

$$\begin{aligned}
\cos\varphi_3(\vartheta) &= \\
\langle v_\vartheta, (-\cos\varphi_1, 0, -\sin\varphi_1\rangle &= -\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2\cos\vartheta \\
&= -\frac{1}{2}\Big(\cos(\varphi_1 - \varphi_2) + \cos(\varphi_1 + \varphi_2) \\
&\quad + \cos\vartheta\,(\cos(\varphi_1 - \varphi_2) - \cos(\varphi_1 + \varphi_2))\Big)
\end{aligned}$$

This is strictly monotone as a function of ϑ on $(0, \pi)$, which shows the uniqueness part of the claim; since the function is continuous, we need only consider the asymptotic behavior.

For $\vartheta \searrow 0$, we obtain

$$\cos\varphi_{3,1} = -\cos(\varphi_1 - \varphi_2) = \cos(\pi - (\varphi_1 - \varphi_2))$$

from which we conclude $\varphi_{3,1} = \pi - (\varphi_1 - \varphi_2) = \varphi_2 + (\pi - \varphi_1) > \varphi_2$.

For $\vartheta \nearrow \pi$, we obtain

$$\cos\varphi_{3,2} = -\cos(\varphi_1 + \varphi_2) = \cos(|\pi - (\varphi_1 + \varphi_2)|)$$

which implies that $\varphi_{3,2} = |\pi - (\varphi_1 + \varphi_2)|$.

Thus, we have shown that $(\varphi_1, \varphi_2, \hat{\varphi}_3)$ is an admissible non-parallel triple if and only if $\hat{\varphi}_3 \in (|\pi - (\varphi_1 + \varphi_2)|, \varphi_2]$.

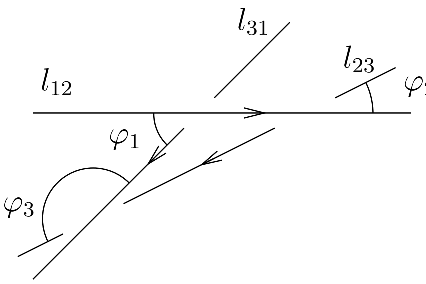
If $\varphi_1 + \varphi_2 > \pi$, this condition is equivalent to III.3.2 (and III.3.1 is clearly satisfied); if $\varphi_1 + \varphi_2 \leq \pi$, this is equivalent to III.3.1 (and III.3.2 holds trivially).

This finishes the proof since the uniqueness statement followed from the strict monotonicity of $\cos\varphi_3(\vartheta)$. □

To give the reader a better imagination of what admissible constellations look like, we present some more sketches (from which one can also read off the asymptotic behavior).

First, we remark that the figures above present the case $\varphi_1 + \varphi_2 < \pi$.

We present a sketch for some small value of ϑ:



In the case $\varphi_1+\varphi_2 > \pi$, we give two slightly different pictures to visualize the asymptotics:

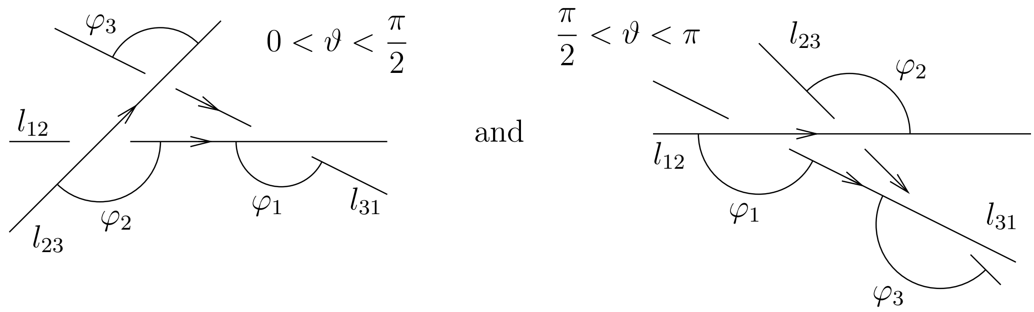


Figure III.4: Constellations with large angle φ_1.

Corollary III.3.4. *A triple* $(\varphi_1, \varphi_2, \varphi_3) \in J^3$ *is a non-parallel admissible triple if and only if* $(r(\varphi_1), r(\varphi_2), r(\varphi_3)) \in \mathcal{T}$. *For every triple of that kind, there are exactly two non-parallel admissible constellations of lines in* $\mathbb{R}^3$ *(modulo* $SO(3)$*).*

Proof. The proof works exactly as the proof of Proposition III.3.3 above:

The difference is just that for angles $\varphi > \pi$, the values of $\tilde{\varphi}^{-1}(\varphi)$ are negative; hence the helicoid $H_{\tilde{\varphi}^{-1}(\varphi)}$ has opposite orientation to the helicoids we have studied in the previous proof.

This has no influence on the proof:

It is still true that for every value of ϑ (notation as in the proof of III.3.3), there is exactly one line that can occur. The function $\cos\varphi_3(\vartheta)$ is not affected, so the conclusion holds (observe that $r(\varphi)$ is the angle in $[0, \pi]$ which occurs between the rays spanning an end). □

For the reader's convenience, we explain in some more detail what the corresponding constellations look like.

If all φ_i are in $(0, \pi) \bigcup_{n\in\mathbb{N}_0}((2n+1)\pi, 2(n+1)\pi)$, all the sketches can remain (almost) the same: The radius of the cylinders C_1, C_2 may be different,

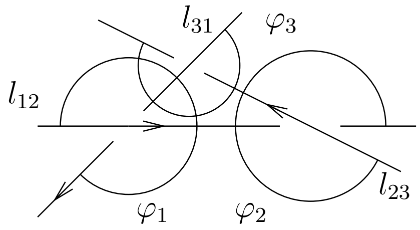


Figure III.5: An admissible constellation with large angles.

but the relative position of the lines remain the same. As an example, we present a sketch of an admissible constellation in the case $\varphi_i \in (\pi, 2\pi)$ for all i in figure III.5.

For angles in $\bigcup_{n\in\mathbb{N}}(2n\pi, (2n+1)\pi)$, we need to be a little more careful: In the sketches, this corresponds essentially to changing which line is above and which one is below.

III.4 Parallel constellations

In this section, we examine for which triples of angles there are parallel constellations.

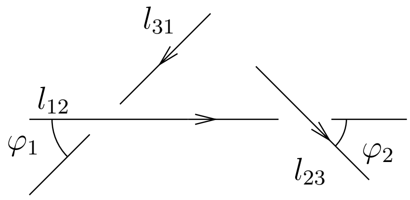
First, let us consider triples $(\varphi_1, \varphi_2, \varphi_3)$ of the form $\pi > \varphi_1 \geq \varphi_2 \geq \varphi_3 > 0$. Since the function h is decreasing on $(0, \pi)$ (compare Lemma III.2.5), we have $h(\varphi_1) \leq h(\varphi_2) \leq h(\varphi_3)$.

The lines are (pairwise) not parallel, so we obtain $h(\varphi_1) + h(\varphi_2) = h(\varphi_3)$.

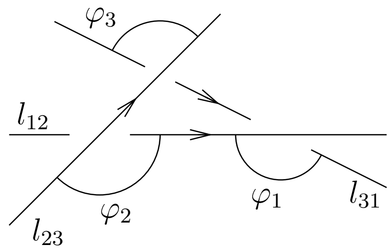
Place l_{12} and l_{31} in $\mathbb{R}^3$ as in the proof of III.3.3.

It follows that l_{23} needs to be contained in $\{y = -h(\varphi_2)\}$.

Considering the orientation of H_{λ_2}, the situation must look as follows:



From the orientation of H_{λ_3}, we conclude that $\varphi_1 + \varphi_2 > \pi$ has to hold, so the constellation of the three lines looks as follows:



Observe that the constellation remains admissible if l_{23} is translated in the plane $\{y = -h(\varphi_2)\}$.

Thus we have shown:

Lemma III.4.1. *A triple of angles with* $\pi > \varphi_1 \geq \varphi_2 \geq \varphi_3 > 0$ *is admissible and corresponds to a parallel admissible constellation if and only if*

$$h(\varphi_1) + h(\varphi_2) = h(\varphi_3) \tag{III.4.1}$$

$$\varphi_1 + \varphi_2 = \pi + \varphi_3 \tag{III.4.2}$$

Similar conditions correspond to the case when the indices of the largest and second largest angles are different.

For such a triple, the corresponding parallel constellations form a one-parameter family. □

Before generalizing to bigger angles, let us determine the admissible parallel triples more explicitely:

Condition III.4.1 translates to

$$\frac{1}{\varphi_1} + \frac{1}{\varphi_2} = \frac{1}{\pi} + \frac{1}{\varphi_3}.$$

Substituting $\varphi_3 = \varphi_1 + \varphi_2 - \pi$, the condition becomes

$$\varphi_1^2 + \varphi_2^2 + \varphi_1\varphi_2 - \pi(\varphi_1 + \varphi_2) = \frac{1}{\pi}.$$

The corresponding level curve is shown in figure III.6.

For bigger angles, we obtain immediately:

Lemma III.4.2. *It is necessary for a triple of angles* $(\varphi_1, \varphi_2, \varphi_3) \in J^3$ *to correspond to a parallel admissible constellation that*

$$h(\varphi_{i_1}) + h(\varphi_{i_2}) = h(\varphi_{i_3}) \tag{III.4.3}$$

$$(r(\varphi_1), r(\varphi_2), r(\varphi_3)) \in \partial\mathcal{T} \tag{III.4.4}$$

where $\{i_1, i_2, i_3\} = \{1, 2, 3\}$ *and* $h(\varphi_{i_3}) \geq h(\varphi_{i_2}) \geq h(\varphi_{i_1})$. □

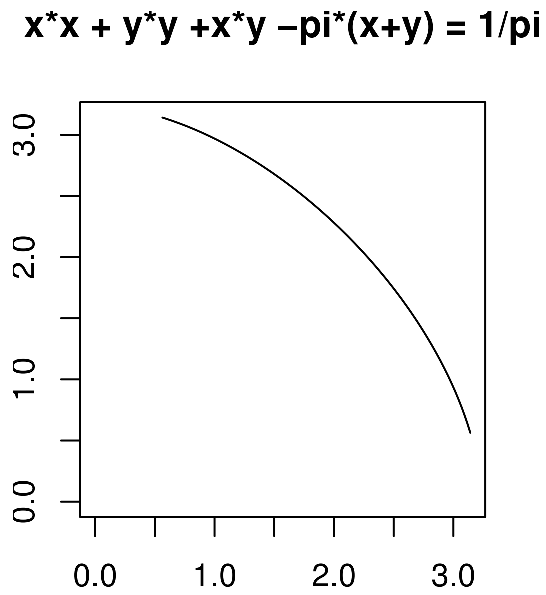


Figure III.6: The level curve $\varphi_1^2 + \varphi_2^2 + \varphi_1\varphi_2 - \pi(\varphi_1 + \varphi_2) = \frac{1}{\pi}$ in the region $[0, \pi]^2$. [4]

For any particular case, it is easy to replace condition III.4.4 by the one corresponding to III.4.1, which yields necessary and sufficient conditions.

Condition III.4.4 implies

Lemma III.4.3. *An admissible triple corresponds* either *to a non-parallel* or *to a parallel constellation.* □

[4]This Plot was made using `R` (http://cran.r-project.org/)

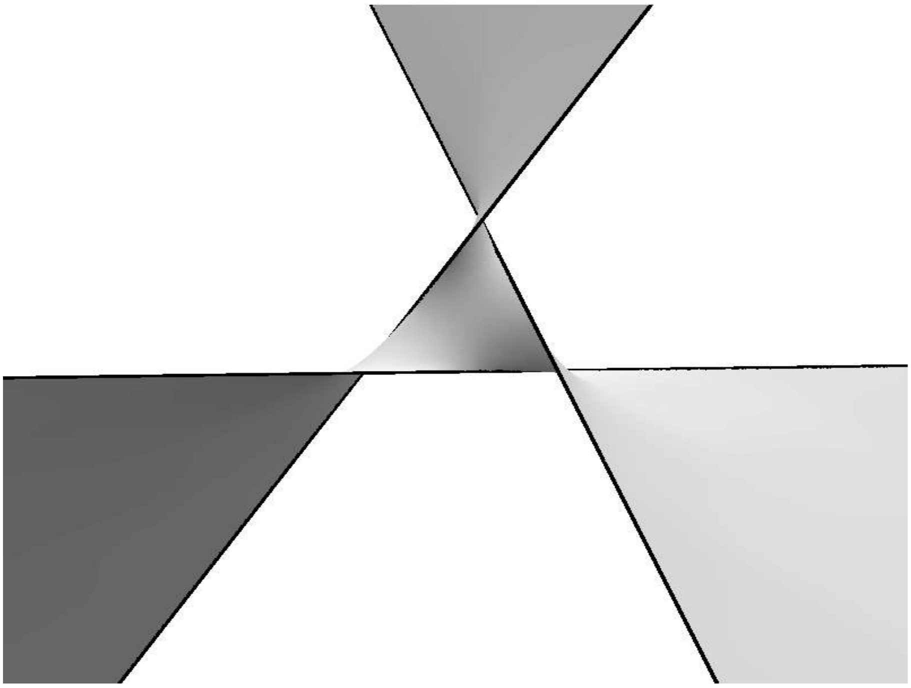


Figure III.7: A minimal surface bounded by three lines (and parts of helicoids)

III.5 Existence of trinoids corresponding to certain admissible constellations

In this section, we show the existence of some trinoids by constructing the corresponding minimal surfaces.

Lemma III.5.1. *A minimal immersed disk Φ bounded by three lines gives rise to a trinoid if and only if its boundary is an admissible constellation and the ends are C^1-asymptotic to the corresponding helicoids. The trinoid M which Φ induces has the property that $\partial\Phi$ is an admissible constellation for $\Psi(M)$.*

Proof. Immediate from definition III.3.1 and Lemma III.2.7. □

One should remark that minimal immersed disks bounded by three lines were already studied by Riemann [Rie61, sec. 17]; more details are given in [Dar87]. However, these papers are hard to read, so we construct some surfaces using the Schwarz reflection principle.

In view of the previous Lemma, we want to construct some trinoids using Theorem I.3.9 (and our corollaries of this Theorem from chapter II).

Theorem III.5.2. *For every $\varphi \in (\pi/3, \pi)$, there exists a trinoid $M_\varphi \in \mathcal{M}$ with $\Psi(M) = (\varphi, \varphi, \varphi)$.*

Proof. According to Proposition III.3.3 and Lemma III.5.1, it suffices to show that there exists a minimal surface corresponding to one of the two admissible constellations for $(\varphi, \varphi, \varphi)$ we found in Proposition III.3.3.

We show that there is a minimal surface with boundary as in the constellation on the left in figure III.3.

We want to show that a solution with maximal symmetry exists. That is: The surface M_φ^c we want to obtain should be invariant under rotation about the bisector of any end. Note that this implies that (part of) the lines bisecting the ends are contained in the surface, and that all ends are isometric.

Take the bisector of one of the ends and join it to the axis of rotational symmetry of the constellation. Join that point to a boundary line along an other bisector.

In this way (cf. figure III.8), we obtain a boundary that can be controlled better.

In fact, Proposition II.3.5 applies and yields a minimal surface with that boundary. Rotating along the boundary lines, we obtain a minimal surface with the required properties (by the Schwarz reflection principle). □

Remark III.5.3. We note that a similar construction does not work for symmetric constellations for angles greater than π: If the angle is greater than 2π, Proposition II.3.5 does not apply any more. If $\varphi \in (\pi, 5\pi/3)$, the Proposition applies. However, one obtains a branch point in the center when trying to extend the surface using Schwarz reflection. ♢

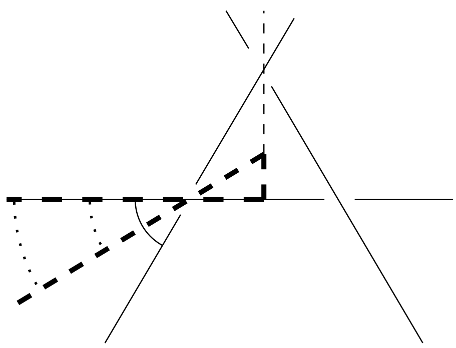


Figure III.8: Cutting along lines of rotational symmetry to obtain a boundary that can be shown to bound an immersed minimal disk.

III.6 Future work

The discussion of trinoids above gives rise to several natural questions, which could not be settled within the scope of this diploma thesis:

- One should examine whether it is possible to use a continuity method to show that for all admissible constellations from $\mathcal{T}$, there exists a corresponding trinoid.
- Also, the question of unicity has been left untouched here. In particular, one has to check whether trinoids corresponding to the second constellation (not used in the construction of trinoids above) exist.
- For the parallel constellations which we have exposed above, it would be interesting to find out whether corresponding trinoids exist. If so, there might be a one-parameter family of trinoids sharing a parameter triple.
- The author is unsure about admissible triples outside of $\mathcal{T}$: Trying to obtain (some of) them with the construction explained above, one obtains minimal surfaces with a branch point. One is tempted to conjecture that this is the case for every minimal disk with prescribed asymptotic behavior of that kind. On the other hand, it seems impossible to obtain the trinoid in the middle of figure III.1 by a continuous deformation of the one on the left; thus, one of the two may have parameter triple $(\varphi, \varphi, \varphi)$ with $\varphi > \pi$.
- Another interesting problem is whether one can understand the meaning of embeddedness resp. Alexandrov-embeddedness for the process of conjugation. Relating CMC-1 surfaces in $\mathbb{R}^3$ to minimal surfaces in S^3, this was obtained and exploited in [GKS01]. Since we could not achieve an understanding of this here, our definition of trinoids is more technical.
- Lastly, it would be interesting to find out whether every properly immersed CMC-1 surface of three ends and genus zero with embedded ends corresponding to distinct boundary points is a trinoid.

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